J. Differential Equations 252 (2012) 6356–6386

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

The Vlasov–Poisson–Boltzmann system in the whole space: The hard potential case

Renjun Duan [a], Tong Yang [b,c], Huijiang Zhao [c,*]

[a] *Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong*
[b] *Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong*
[c] *School of Mathematics and Statistics, Wuhan University, PR China*

ARTICLE INFO

Article history:
Received 31 October 2011
Available online 6 April 2012

ABSTRACT

This paper is concerned with the Cauchy problem on the Vlasov–Poisson–Boltzmann system for hard potentials in the whole space. When the initial data is a small perturbation of a global Maxwellian, a satisfactory global existence theory of classical solutions to this problem, together with the corresponding temporal decay estimates on the global solutions, is established. Our analysis is based on time-decay properties of solutions and a new time–velocity weight function which is designed to control the large-velocity growth in the nonlinear term for the case of non-hard-sphere interactions.

Contents

* Corresponding author.
E-mail addresses: rjduan@math.cuhk.edu.hk (R. Duan), matyang@cityu.edu.hk (T. Yang), hhjjzhao@hotmail.com (H. Zhao).

0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2012.03.012

1. Introduction

The Vlasov–Poisson–Boltzmann (called VPB in the sequel for simplicity) system is a physical model describing mutual interactions of the electrons through collisions in the self-consistent electric field. When the constant background charge density is normalized to be unit, the VPB system takes the form of

$$\partial_t f + \xi \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_\xi f = Q(f, f), \tag{1.1}$$

$$\Delta_x \phi = \int_{\mathbb{R}^3} f \, d\xi - 1, \qquad \phi(x) \to 0 \quad \text{as } |x| \to \infty, \tag{1.2}$$

$$f(0, x, \xi) = f_0(x, \xi). \tag{1.3}$$

Here the unknown $f = f(t, x, \xi) \geqslant 0$ is the number density for the particles located at $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ at time $t \geqslant 0$. The potential function $\phi = \phi(t, x)$ generating the self-consistent electric field $\nabla_x \phi$ in (1.1) is coupled with $f(t, x, \xi)$ through the Poisson equation (1.2). The bilinear collision operator Q acting only on the velocity variable [2,11] is defined by

$$Q(f, g) = \iint_{\mathbb{R}^3 \times S^2} |\xi - \xi_*|^\gamma q_0(\theta) \big\{ f\big(\xi_*'\big) g\big(\xi'\big) - f(\xi_*) g(\xi) \big\} \, d\omega \, d\xi_*.$$

Here

$$\xi' = \xi - \big[(\xi - \xi_*) \cdot \omega\big]\omega, \qquad \xi_*' = \xi_* + \big[(\xi - \xi_*) \cdot \omega\big]\omega, \quad \omega \in S^2,$$

is the relation between velocities ξ', ξ_*' after and the velocities ξ, ξ_* before the collision, which is induced by the conservation of momentum and energy.

Throughout this paper, we are concentrated on the hard potential case, i.e., $0 \leqslant \gamma \leqslant 1$ under Grad's angular cutoff assumption

$$0 \leqslant q_0(\theta) \leqslant C|\cos\theta|, \quad \cos\theta = (\xi - \xi_*)/|\xi - \xi_*| \cdot \omega.$$

The case for soft potentials will be pursued by the same authors in a forthcoming manuscript [10].

Let $\mathbf{M} = (2\pi)^{-3/2} e^{-|\xi|^2/2}$ be a normalized Maxwellian. We are concerned with the well-posedness of the Cauchy problem (1.1)–(1.3) when f_0 is sufficiently close to $\mathbf{M}$ in a certain sense that we shall clarify later on. To this end, set the perturbation u by $f - \mathbf{M} = \mathbf{M}^{1/2} u$, then, the Cauchy problem (1.1)–(1.3) of the VPB system is reformulated as

$$\partial_t u + \xi \cdot \nabla_x u + \nabla_x \phi \cdot \nabla_\xi u - \frac{1}{2} \xi \cdot \nabla_x \phi u - \nabla_x \phi \cdot \xi \mathbf{M}^{1/2} = \mathbf{L} u + \Gamma(u, u), \tag{1.4}$$

$$\Delta_x \phi = \int_{\mathbb{R}^3} \mathbf{M}^{1/2} u \, d\xi, \qquad \phi(x) \to 0 \quad \text{as } |x| \to \infty, \tag{1.5}$$

$$u(0, x, \xi) = u_0(x, \xi) = \mathbf{M}^{-1/2}(f_0 - \mathbf{M}), \tag{1.6}$$

where

$$\mathbf{L}u = \mathbf{M}^{-\frac{1}{2}}\left\{Q\left(\mathbf{M}, \mathbf{M}^{1/2}u\right) + Q\left(\mathbf{M}^{1/2}u, \mathbf{M}\right)\right\},$$
$$\Gamma(u,u) = \mathbf{M}^{-\frac{1}{2}}Q\left(\mathbf{M}^{1/2}u, \mathbf{M}^{1/2}u\right).$$

Note that one can write $\mathbf{L} = -\nu + K$ with $\nu = \nu(\xi) \sim (1+|\xi|)^\gamma$ and $Ku = \int_{\mathbb{R}^3} K(\xi,\xi_*)u(\xi_*)\,d\xi_*$ for a real symmetric integral kernel $K(\xi,\xi_*)$; see [11, Section 3.2]. In addition, due to (1.5), ϕ is always determined in terms of u by

$$\phi(t,x) = -\frac{1}{4\pi|x|} *_x \int_{\mathbb{R}^3} \mathbf{M}^{1/2}u(t,x,\xi)\,d\xi.$$

Observe that by plugging the above formula into the dynamical equation (1.4) of the reformulated VPB system, one has the single evolution equation for the perturbation u; see [9,8].

Before stating our main result, we first introduce a mixed time–velocity weight function

$$w_\ell(t,\xi) = \langle\xi\rangle^{\frac{\ell}{2}} e^{\frac{\lambda|\xi|}{(1+t)^\theta}}, \tag{1.7}$$

where $\ell \in \mathbb{R}$, $\lambda > 0$ and $\theta > 0$ are suitably chosen constants, and $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$. For given $u(t,x,\xi)$ and an integer N, define a temporal energy norm

$$|\!|\!|u|\!|\!|_{N,\ell}(t) = \sum_{|\alpha|+|\beta|\leqslant N} \left\|w_\ell(t,\xi)\partial^\alpha_\beta u(t)\right\| + \left\|\nabla_x\phi(t)\right\|_{H^N}, \tag{1.8}$$

where as in [6,9], time derivatives are not included into the energy norm.

The main result of this paper is stated as follows. Notations will be explained at the end of this section.

Theorem 1.1. *Let $N \geqslant 4$, $\ell \geqslant 2$, $\lambda > 0$, $0 < \theta \leqslant 1/4$. Assume that $f_0 = \mathbf{M} + \mathbf{M}^{1/2}u_0 \geqslant 0$ and $\iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{M}^{1/2}u_0\,dx\,d\xi = 0$, then there exist positive constants $\epsilon_0 > 0$, $C > 0$ such that if*

$$\sum_{|\alpha|+|\beta|\leqslant N} \left\|w_\ell(0,\xi)\partial^\alpha_\beta u_0\right\| + \left\|\left(1+|x|\right)u_0\right\|_{Z_1} \leqslant \epsilon_0, \tag{1.9}$$

the Cauchy problem (1.4), (1.5), (1.6) *of the VPB system admits a unique global solution $u(t,x,\xi)$ satisfying $f(t,x,\xi) = \mathbf{M} + \mathbf{M}^{1/2}u(t,x,\xi) \geqslant 0$ and*

$$\sup_{t\geqslant 0}\left\{(1+t)^{\frac{3}{4}}|\!|\!|u|\!|\!|_{N,\ell}(t)\right\} \leqslant C\epsilon_0. \tag{1.10}$$

Under the framework of small perturbations around global Maxwellians either in the whole space or on torus, there have been extensive studies on the VPB system [16,30,31,9,8,32,29] and even the more general Vlasov–Maxwell–Boltzmann system [15,24,7,5]. However, only the hard-sphere model with $\gamma = 1$ is considered among those existing works, and the case of general hard potentials $0 \leqslant \gamma < 1$ has remained open. One of the main difficulties lies in the fact that the dissipation of the linearized Boltzmann operator $\mathbf{L}$ for non-hard-sphere potentials $\gamma < 1$ cannot control the full nonlinear dynamics due to the velocity growth effect of $\xi \cdot \nabla_x\phi u$ in the nonlinear term. One of our main ideas is to introduce the new mixed time–velocity weight function $w_\ell(t,\xi)$, especially the factor $\exp\{\lambda|\xi|/(1+t)^\theta\}$, to overcome this main difficulty and the main purpose of this paper is to show that a suitable application of such a new weight can indeed yield a satisfactory global existence theory of classical solution to the VPB system in the whole space for the case $0 \leqslant \gamma \leqslant 1$. It is worth to pointing out that the arguments employed here can be adopted straightforwardly to deal with either the VPB system on torus with additional conservation laws as in [16] or the two-species VPB system as in [15,32,29].

The proof of Theorem 1.1 is based on a new weighted energy method. Before explaining the feature of this method, let us recall some existing work related to Theorem 1.1. In the perturbation theory of the Boltzmann equation for the global well-posedness of solutions around Maxwellians, the energy method was first developed independently in [16,14] and in [22,21]. We also mention the pioneering work [26] and its recent improvement [27] by using the spectral analysis and the contraction mapping principle. When the self-induced potential force is taken into account, even though [13] considered the spectral property of the system, the spectral theory corresponding to [26] has not been known so far, partially because the Poisson equation produces an additional nonlocal term with singular kernels.

Fortunately, the energy method still works well in the presence of the self-induced electric field [16,9] or even electromagnetic field [15,24,5]. Moreover, in such situations, the large-time behavior of global solutions is also extensively studied in recent years by using different approaches. One approach which usually leads to slower time-decay than in the linearized level is used in [31] on the basis of the improved energy estimates together with functional inequalities. The method of thirteen moments and compensation functions is found by [19] which gives the optimal time rate without using the spectral theory; see [12] and [29] for two applications. Recently, concerning with the optimal time rate, a time–frequency analysis method has been developed in [8,7,5]. Precisely, in the same spirit of [28], some time–frequency functionals or interactive energy functionals are constructed in [8,7,5] to capture the dissipation of the degenerate components of the full system. We finally also mention [25] about the time–velocity splitting method for the study of soft potentials. It would be quite interesting to combine [25] with the current work to investigate the same topic for the VPB system with soft potentials $-3 < \gamma < 0$, which is now under our current research [10].

Our weighted energy method used here contains some new ingredients, compared with the previous work [16] for the hard-sphere model. One of the most important ingredients is to combine the time-decay of solutions with the usual weighted energy inequalities in order to obtain the uniform-in-time *a priori* estimates. In fact, the pure energy estimates without using time-decay cannot be closed; see (4.7). As mentioned before, this is because the nonlinear term $\xi \cdot \nabla_x \phi u$ may increase linearly in $|\xi|$ but the dissipation of systems only has the growth of $|\xi|^\gamma$ with $0 \leqslant \gamma \leqslant 1$ as given in (2.1). Therefore, we are forced to postulate the *a priori* assumption (**A1**) on the time-decay of solutions with certain explicit rates so that the trouble term $\xi \cdot \nabla_x \phi u$ can be controlled through introducing the mixed time–velocity weight factor $\exp\{\lambda|\xi|/(1+t)^\theta\}$; see the key estimates (4.16), (4.19) and (4.23). Formally, a good term with the extra weight in the form of $|\xi|/(1+t)^{1+\theta}$ naturally arises from the time derivative of $\exp\{\lambda|\xi|/(1+t)^\theta\}$ in the $w_\ell(t,\xi)$-weighted estimate. For all details of these arguments, see the proof of Lemmas 4.2 and 4.3. We here remark that Lemmas 2.1, 2.2 and 2.3 play a vital role in the weighted energy estimates. In particular, we used a velocity–time splitting trick to prove (2.9) in Lemma 2.2 concerning the $w_\ell(t,\xi)$-weighted estimate on K. An important observation is that the second part on the right-hand side of (2.9) not only excludes the exponential weight factor $\exp\{\lambda|\xi|/(1+t)^\theta\}$ but also contains the strictly slower velocity growth $\langle\xi\rangle^{\ell-1}$ than $\langle\xi\rangle^{\ell}$.

To recover the time-decay of solutions, we apply the Duhamel's principle to the nonlinear system and use the linearized time-decay property combined with two nonlinear energy estimates (5.3) and (4.27) under both *a priori* assumptions (**A1**) and (**A2**). Thus, the uniform-in-time *a priori* estimates can be closed with the help of the time-weighted energy functional $X_{N,\ell}(t)$ given in (5.1). It should be emphasized that $\|\nabla_x^2\phi\|_{H^{N-1}}$ decays with the rate at most $(1+t)^{-5/4}$ and hence we have to assume $0 < \theta \leqslant 1/4$ in the weight function $w_\ell(t,\xi)$. Exactly the high-order energy functional $\mathcal{E}^{\mathrm{h}}_{N,\ell}(t)$ is employed to obtain the time-decay of $\|\nabla_x^2\phi\|_{H^{N-1}}$. In addition, we also point out that the condition $\ell \geqslant 2$ in $w_\ell(t,\xi)$ is necessary in our proof, otherwise one cannot use the energy functional $\mathcal{E}_{N,\ell}(t)$ to bound the full nonlinear term; see (5.7) and (5.13).

Finally, for the study of the VPB system in other respects, we also mention [1,3,17,20,23]. Notice that [20] used the approach of the well-known work [4] to establish the trend of solutions to global Maxwellians for the VPB system with general potentials in the collision kernel but under some additional conditions.

The rest of this paper is organized as follows. In Section 2, we mainly prove some important lemmas to show how the new weight function $w_\ell(t,\xi)$ is involved in the estimates with the linear operator $L = -\nu + K$ and the nonlinear operator $\Gamma(\cdot,\cdot)$. In Section 3, we improve the linearized result [8, Theorem 2] under the neutral assumption $\int_{\mathbb{R}^3} a_0(x)\,dx = 0$. In Section 4, we devote ourselves to the *a priori* estimates, and in Section 5, we complete the proof of Theorem 1.1.

Notations. Throughout this paper, C denotes some positive (generally large) constant and κ denotes some positive (generally small) constant, where both C and κ may take different values in different places. $A \sim B$ means $\kappa A \leqslant B \leqslant \frac{1}{\kappa} A$ for a generic constant $0 < \kappa < 1$. For an integer $m \geqslant 0$, we use $H^m_{x,\xi}$, H^m_x, H^m_ξ to denote the usual Hilbert spaces $H^m(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$, $H^m(\mathbb{R}^3_x)$, $H^m(\mathbb{R}^3_\xi)$, respectively, and L^2, L^2_x, L^2_ξ are used for the case when $m = 0$. When without confusion, we use H^m to denote H^m_x and use L^2 to denote L^2_x or $L^2_{x,\xi}$. For $q \geqslant 1$, we also define the mixed velocity–space Lebesgue space $Z_q = L^2_\xi(L^q_x) = L^2(\mathbb{R}^3_\xi; L^q(\mathbb{R}^3_x))$ with the norm

$$\|u\|_{Z_q} = \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \left|u(x,\xi)\right|^q dx \right)^{2/q} d\xi \right)^{1/2}, \quad u = u(x,\xi) \in Z_q.$$

For multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$, we denote $\partial^\alpha_\beta = \partial^\alpha_x \partial^\beta_\xi$, that is, $\partial^\alpha_\beta = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \partial^{\alpha_3}_{x_3} \partial^{\beta_1}_{\xi_1} \partial^{\beta_2}_{\xi_2} \partial^{\beta_3}_{\xi_3}$. The length of α is $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and the length of β is $|\beta| = \beta_1 + \beta_2 + \beta_3$.

2. Preliminaries

It is known that the linearized collision operator $\mathbf{L}$ is non-positive, the null space of $\mathbf{L}$ is given by

$$\mathcal{N} = \operatorname{span}\left\{\mathbf{M}^{1/2}, \xi_i \mathbf{M}^{1/2}\ (1 \leqslant i \leqslant 3), |\xi|^2 \mathbf{M}^{1/2}\right\},$$

and $-\mathbf{L}$ is locally coercive in the sense that there is a constant $\kappa_0 > 0$ such that [2]

$$-\int_{\mathbb{R}^3} u \mathbf{L} u \, d\xi \geqslant \kappa_0 \int_{\mathbb{R}^3} \nu(\xi) \left|\{\mathbf{I} - \mathbf{P}\} u\right|^2 d\xi \tag{2.1}$$

holds for $u = u(\xi)$, where $\mathbf{I}$ means the identity operator and $\mathbf{P}$ denotes the orthogonal projection from L^2_ξ to $\mathcal{N}$. Given any $u(t,x,\xi)$, one can write $\mathbf{P}$ in (2.1) as

$$\begin{cases} \mathbf{P}u = \left\{a(t,x) + b(t,x)\cdot\xi + c(t,x)\left(|\xi|^2 - 3\right)\right\}\mathbf{M}^{1/2}, \\ a = \displaystyle\int_{\mathbb{R}^3} \mathbf{M}^{1/2} u \, d\xi = \int_{\mathbb{R}^3} \mathbf{M}^{1/2} \mathbf{P} u \, d\xi, \\ b_i = \displaystyle\int_{\mathbb{R}^3} \xi_i \mathbf{M}^{1/2} u \, d\xi = \int_{\mathbb{R}^3} \xi_i \mathbf{M}^{1/2} \mathbf{P} u \, d\xi, \quad 1 \leqslant i \leqslant 3, \\ c = \displaystyle\frac{1}{6}\int_{\mathbb{R}^3} \left(|\xi|^2 - 3\right)\mathbf{M}^{1/2} u \, d\xi = \frac{1}{6}\int_{\mathbb{R}^3} \left(|\xi|^2 - 3\right)\mathbf{M}^{1/2} \mathbf{P} u \, d\xi, \end{cases}$$

so that

$$u(t,x,\xi) = \mathbf{P}u(t,x,\xi) + \{\mathbf{I} - \mathbf{P}\}u(t,x,\xi).$$

Here, $\mathbf{P}u$ is called the macroscopic component of $u(t,x,\xi)$ and $\{\mathbf{I}-\mathbf{P}\}u$ the microscopic component of $u(t,x,\xi)$, cf. [16,15,14] and [22,21]. For later use, one can rewrite $\mathbf{P}$ as

$$\begin{cases} \mathbf{P}u = \mathbf{P}_0 u \oplus \mathbf{P}_1 u, \\ \mathbf{P}_0 u = a(t,x)\mathbf{M}^{1/2}, \\ \mathbf{P}_1 u = \{b(t,x)\cdot\xi + c(t,x)(|\xi|^2-3)\}\mathbf{M}^{1/2}, \end{cases} \tag{2.2}$$

where $\mathbf{P}_0$ and $\mathbf{P}_1$ are the projectors corresponding to the hyperbolic and parabolic parts of the macroscopic component, respectively, cf. [8].

Recall that $\mathbf{L} = -\nu + K$ is defined as

$$\nu(\xi) = \iint\limits_{\mathbb{R}^3\times S^2} |\xi-\xi_*|^\gamma q_0(\theta)\mathbf{M}(\xi_*)\,d\omega\,d\xi_* \sim \left(1+|\xi|\right)^\gamma, \tag{2.3}$$

and

$$\begin{aligned} Ku(\xi) = {} & \iint\limits_{\mathbb{R}^3\times S^2} |\xi-\xi_*|^\gamma q_0(\theta)\mathbf{M}^{1/2}(\xi_*)\mathbf{M}^{1/2}(\xi'_*)u(\xi')\,d\omega\,d\xi_* \\ & + \iint\limits_{\mathbb{R}^3\times S^2} |\xi-\xi_*|^\gamma q_0(\theta)\mathbf{M}^{1/2}(\xi_*)\mathbf{M}^{1/2}(\xi')u(\xi'_*)\,d\omega\,d\xi_* \\ & - \iint\limits_{\mathbb{R}^3\times S^2} |\xi-\xi_*|^\gamma q_0(\theta)\mathbf{M}^{1/2}(\xi_*)\mathbf{M}^{1/2}(\xi)u(\xi_*)\,d\omega\,d\xi_* \\ = {} & \int\limits_{\mathbb{R}^3} K(\xi,\xi_*)u(\xi_*)\,d\xi_*. \end{aligned} \tag{2.4}$$

For properties on the collision frequency $\nu(\xi)$ and the integral operator K, we have

Lemma 2.1.

(i) *$\nu(\xi)$ is smooth in ξ, and for $\beta > 0$, $\partial_\beta \nu(\xi)$ is bounded.*
(ii) *Let $\beta \geqslant 0$. For any $0 < q < 1$, there is $C_{|\beta|,q}$ such that*

$$|\partial_\beta K u| \leqslant C_{|\beta|,q} \int\limits_{\mathbb{R}^3} K_q(\xi,\xi_*) \sum_{|\beta'|\leqslant|\beta|} \left|\partial_{\beta'} u(\xi_*)\right| d\xi_*, \tag{2.5}$$

where $K_q(\xi,\xi_*)$ is a real nonnegative symmetric kernel in the form of

$$K_q(\xi,\xi_*) = \left\{|\xi-\xi_*| + |\xi-\xi_*|^{-1}\right\} e^{-\frac{q}{8}|\xi-\xi_*|^2 - \frac{q}{8}\frac{||\xi|^2-|\xi_*|^2|^2}{|\xi-\xi_*|^2}}. \tag{2.6}$$

Moreover, for any $\eta > 0$, there is $C_{|\beta|,\eta} > 0$ such that

$$\|\partial_\beta K u\| \leqslant \eta \sum_{|\beta'|=|\beta|} \|\partial_{\beta'} u\| + C_{|\beta|,\eta}\|u\|. \tag{2.7}$$

Proof. Write (2.3) as

$$\nu(\xi) = C_{q_0} \int_{\mathbb{R}^3} |z|^\gamma \mathbf{M}(\xi + z)\, dz$$

for some constant $C_{q_0} > 0$. Since $\gamma \geqslant 0$ and $\mathbf{M}(\xi)$ is smooth and decays exponentially in ξ, $\nu(\xi)$ is smooth in ξ. For $\beta > 0$, write $\beta = \beta' + \beta_i$ with $|\beta_i| = 1$. Note that $\partial_i |z|^\gamma = \gamma |z|^{\gamma-1} z_i/|z|$ whenever $z \neq 0$. Then, from integration by parts,

$$\partial_\beta \nu(\xi) = C_{q_0} \int_{\mathbb{R}^3} |z|^\gamma \partial_\beta \mathbf{M}(\xi + z)\, dz = C_{q_0} \int_{\mathbb{R}^3} \gamma |z|^{\gamma-1} \frac{z_i}{|z|} \partial_{\beta'} \mathbf{M}(\xi + z)\, dz.$$

Since $|\partial_{\beta'} \mathbf{M}(\xi + z)| \leqslant C\mathbf{M}^{q'}(\xi + z)$ for some $0 < q' < 1$,

$$\left|\partial_\beta \nu(\xi)\right| \leqslant C \int_{\mathbb{R}^3} |z|^{\gamma-1} \mathbf{M}^{q'}(\xi + z)\, dz.$$

Therefore, $\partial_\beta \nu(\xi)$ is bounded due to $0 \leqslant \gamma \leqslant 1$. Then (i) is proved.

To prove (ii), rewrite (2.4) as

$$\begin{aligned} Ku(\xi) = {} & \iint_{\mathbb{R}^3 \times S^2} |z|^\gamma q_0(\theta) \mathbf{M}^{1/2}(\xi + z) \mathbf{M}^{1/2}(\xi + z_\perp) u(\xi + z_\parallel)\, d\omega\, dz \\ & + \iint_{\mathbb{R}^3 \times S^2} |z|^\gamma q_0(\theta) \mathbf{M}^{1/2}(\xi + z) \mathbf{M}^{1/2}(\xi + z_\parallel) u(\xi - z_\perp)\, d\omega\, dz \\ & - \iint_{\mathbb{R}^3 \times S^2} |z|^\gamma q_0(\theta) \mathbf{M}^{1/2}(\xi + z) \mathbf{M}^{1/2}(\xi) u(\xi + z)\, d\omega\, dz, \end{aligned}$$

where for given $\omega \in S^2$, $z_\parallel = z \cdot \omega\omega$ and $z_\perp = z - z_\parallel$. Then, for $\beta \geqslant 0$, $\partial_\beta Ku(\xi)$ takes the form of

$$\begin{aligned} & \sum_{\beta' \leqslant \beta} C_{\beta'}^\beta \iint_{\mathbb{R}^3 \times S^2} |z|^\gamma q_0(\theta) \partial_{\beta-\beta'} \big\{\mathbf{M}^{1/2}(\xi + z) \mathbf{M}^{1/2}(\xi + z_\perp)\big\} \partial_{\beta'} u(\xi + z_\parallel)\, d\omega\, dz \\ & \quad + \sum_{\beta' \leqslant \beta} C_{\beta'}^\beta \iint_{\mathbb{R}^3 \times S^2} |z|^\gamma q_0(\theta) \partial_{\beta-\beta'} \big\{\mathbf{M}^{1/2}(\xi + z) \mathbf{M}^{1/2}(\xi + z_\parallel)\big\} \partial_{\beta'} u(\xi - z_\perp)\, d\omega\, dz \\ & \quad - \sum_{\beta' \leqslant \beta} C_{\beta'}^\beta \iint_{\mathbb{R}^3 \times S^2} |z|^\gamma q_0(\theta) \partial_{\beta-\beta'} \big\{\mathbf{M}^{1/2}(\xi + z) \mathbf{M}^{1/2}(\xi)\big\} \partial_{\beta'} u(\xi + z)\, d\omega\, dz. \end{aligned}$$

Here observe that for any $0 < q < 1$, there is $C_{|\beta|,q}$ such that

$$\begin{aligned} \left|\partial_{\beta-\beta'} \big\{\mathbf{M}^{1/2}(\xi + z) \mathbf{M}^{1/2}(\xi + z_\perp)\big\}\right| &\leqslant C_{|\beta|,q} \mathbf{M}^{q/2}(\xi + z) \mathbf{M}^{q/2}(\xi + z_\perp), \\ \left|\partial_{\beta-\beta'} \big\{\mathbf{M}^{1/2}(\xi + z) \mathbf{M}^{1/2}(\xi + z_\parallel)\big\}\right| &\leqslant C_{|\beta|,q} \mathbf{M}^{q/2}(\xi + z) \mathbf{M}^{q/2}(\xi + z_\parallel), \\ \left|\partial_{\beta-\beta'} \big\{\mathbf{M}^{1/2}(\xi + z) \mathbf{M}^{1/2}(\xi)\big\}\right| &\leqslant C_{|\beta|,q} \mathbf{M}^{q/2}(\xi + z) \mathbf{M}^{q/2}(\xi). \end{aligned}$$

Thus, $|\partial_\beta Ku(\xi)|$ is bounded by

$$C_{|\beta|,q}\sum_{\beta'\leqslant\beta}\left\{\iint\limits_{\mathbb{R}^3\times S^2}|z|^\gamma q_0(\theta)\mathbf{M}^{q/2}(\xi+z)\mathbf{M}^{q/2}(\xi+z_\perp)\left|\partial_{\beta'}u(\xi+z_\parallel)\right|d\omega\,dz\right.$$
$$+\iint\limits_{\mathbb{R}^3\times S^2}|z|^\gamma q_0(\theta)\mathbf{M}^{q/2}(\xi+z)\mathbf{M}^{q/2}(\xi+z_\parallel)\left|\partial_{\beta'}u(\xi-z_\perp)\right|d\omega\,dz$$
$$\left.+\iint\limits_{\mathbb{R}^3\times S^2}|z|^\gamma q_0(\theta)\mathbf{M}^{q/2}(\xi+z)\mathbf{M}^{q/2}(\xi)\left|\partial_{\beta'}u(\xi+z)\right|d\omega\,dz\right\}.$$

From the above expression of the upper bound of $|\partial_\beta Ku(\xi)|$, one can now use the same calculations as in [11, Section 3.2] to obtain (2.5) with the integral kernel $K_q(\xi,\xi_*)$ defined by (2.6). Furthermore, by also the same proof as in [11, Section 3.5], (2.5) together with (2.6) implies that K is a compact operator from $H^{|\beta|}_\xi$ to $H^{|\beta|}_\xi$, which leads to (2.7). (ii) is proved. □

Now we state two weighted estimates on the integral operator K and the nonlinear term $\Gamma(f,g)$ with respect to the new time–velocity weight $w_\ell(t,\xi)$. First for the weighted estimates on the integral operator K, we have

Lemma 2.2. *Let $\ell\in\mathbb{R}$, $0<q<1$ and let $\beta\geqslant 0$ be a multi-index. Then, there is $C_{q,\ell}>0$ such that*

$$\int\limits_{\mathbb{R}^3}K_q(\xi,\xi_*)\frac{w_\ell(t,\xi)}{w_\ell(t,\xi_*)}\,d\xi_*\leqslant\frac{C_{q,\ell}}{1+|\xi|}. \tag{2.8}$$

Moreover, for any $\eta>0$, there is $C_{|\beta|,\ell,\eta}>0$ such that

$$\left|\int\limits_{\mathbb{R}^3}w_\ell^2(t,\xi)\nu\,\partial_\beta Ku\,d\xi\right|\leqslant\eta\int\limits_{\mathbb{R}^3}w_\ell^2(t,\xi)\left\{|\nu|^2+\sum_{|\beta'|\leqslant|\beta|}|\partial_{\beta'}u|^2\right\}d\xi$$
$$+C_{|\beta|,\ell,\eta}\int\limits_{\mathbb{R}^3}\langle\xi\rangle^{\ell-1}\left\{|\nu|^2+\sum_{|\beta'|\leqslant|\beta|}|\partial_{\beta'}u|^2\right\}d\xi. \tag{2.9}$$

Proof. Notice

$$\frac{w_\ell(t,\xi)}{w_\ell(t,\xi_*)}=\frac{\langle\xi\rangle^{\frac{\ell}{2}}e^{\frac{\lambda|\xi|}{(1+t)^\theta}}}{\langle\xi_*\rangle^{\frac{\ell}{2}}e^{\frac{\lambda|\xi_*|}{(1+t)^\theta}}}$$
$$\leqslant C_\ell\langle\xi-\xi_*\rangle^{\frac{|\ell|}{2}}e^{\frac{\lambda|\xi-\xi_*|}{(1+t)^\theta}}\leqslant C_\ell\langle\xi-\xi_*\rangle^{\frac{|\ell|}{2}}e^{\epsilon|\xi-\xi_*|^2+\frac{\lambda^2}{4\epsilon}}$$

for any $\epsilon>0$. Recall (2.6) for $K_q(\xi,\xi_*)$, we have by fixing $\epsilon>0$ small enough that

$$K_q(\xi,\xi_*)\frac{w_\ell(t,\xi)}{w_\ell(t,\xi_*)}\leqslant C_{\ell,\lambda}K_{q'}(\xi,\xi_*)$$

holds for some $0 < q' < q$. Hence, by using

$$\int_{\mathbb{R}^3} K_{q'}(\xi,\xi_*)\langle\xi_*\rangle^{-s}\,d\xi_* \leqslant C_{q'}\langle\xi\rangle^{-(s+1)}$$

for any $s \geqslant 0$ as proved in [11, Lemma 3.3.1], (2.8) follows. To prove (2.9), by applying (2.5), one has

$$\left|\int_{\mathbb{R}^3} w_\ell^2(t,\xi)\nu\partial_\beta K u\,d\xi\right| \leqslant C\sum_{|\beta'|\leqslant|\beta|}\iint_{\mathbb{R}^3\times\mathbb{R}^3} w_\ell^2(t,\xi)\big|\nu(\xi)\big|K_q(\xi,\xi_*)\big|\partial_{\beta'}u(\xi_*)\big|\,d\xi\,d\xi_*,$$

where from Hölder's inequality, each term in the right-hand summation is further bounded by

$$\begin{aligned}&\left\{\iint_{\mathbb{R}^3\times\mathbb{R}^3} K_q(\xi,\xi_*)\frac{w_\ell^2(t,\xi)}{w_\ell^2(t,\xi_*)}\big|w_\ell(t,\xi)\nu(\xi)\big|^2\,d\xi\,d\xi_*\right\}^{\frac12}\\&\quad\times\left\{\iint_{\mathbb{R}^3\times\mathbb{R}^3} K_q(\xi,\xi_*)\big|w_\ell(t,\xi_*)\partial_{\beta'}u(\xi_*)\big|^2\,d\xi\,d\xi_*\right\}^{\frac12}.\end{aligned}$$

Therefore, by (2.8), it follows that

$$\left|\int_{\mathbb{R}^3} w_\ell^2(t,\xi)\nu\partial_\beta K u\,d\xi\right| \leqslant C\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{w_\ell^2(t,\xi)}{1+|\xi|}\left\{\big|\nu(\xi)\big|^2+\sum_{|\beta'|\leqslant|\beta|}\big|\partial_{\beta'}u(\xi)\big|^2\right\}d\xi\,d\xi_*. \tag{2.10}$$

We now split the integration domain into $|\xi| \geqslant R(1+t)^\theta$ and $|\xi| \leqslant R(1+t)^\theta$ for an arbitrary constant $R > 0$. For the case when $|\xi| \geqslant R(1+t)^\theta$, due to $\theta > 0$, one has $|\xi| \geqslant R$, and hence

$$\frac{w_\ell^2(t,\xi)}{1+|\xi|}\chi_{|\xi|\geqslant R(1+t)^\theta} \leqslant \frac{1}{1+R}w_\ell^2(t,\xi),$$

while if $|\xi| \leqslant R(1+t)^\theta$ one has

$$\frac{w_\ell^2(t,\xi)}{1+|\xi|}\chi_{|\xi|\leqslant R(1+t)^\theta} = \frac{\langle\xi\rangle^\ell e^{\frac{2\lambda|\xi|}{(1+t)^\theta}}}{1+|\xi|}\chi_{|\xi|\leqslant R(1+t)^\theta} \leqslant C\langle\xi\rangle^{\ell-1}e^{2\lambda R}.$$

Then, (2.9) follows by applying the above two estimates into (2.10). This completes the proof of Lemma 2.2. □

For the weighted estimates on the nonlinear term $\Gamma(f,g)$, we have

Lemma 2.3. *Let $\ell \geqslant 0$, and let $\beta \geqslant 0$ be a multi-index.*

$$\left|\int_{\mathbb{R}^3} w_\ell^2(t,\xi)h(\xi)\partial_\beta\Gamma(f,g)\,d\xi\right|$$

$$\leqslant C \sum_{\beta_1+\beta_2\leqslant\beta} \Biggl\{ \Biggl[\int_{\mathbb{R}^3} \nu(\xi) w_\ell^2(t,\xi) \bigl|\partial_{\beta_1} f(\xi)\bigr|^2 \, d\xi \Biggr]^{\frac{1}{2}} \Biggl[\int_{\mathbb{R}^3} w_0^2(t,\xi) \bigl|\partial_{\beta_2} g(\xi)\bigr|^2 \, d\xi \Biggr]^{\frac{1}{2}}$$

$$+ \Biggl[\int_{\mathbb{R}^3} \nu(\xi) w_\ell^2(t,\xi) \bigl|\partial_{\beta_2} g(\xi)\bigr|^2 \, d\xi \Biggr]^{\frac{1}{2}} \Biggl[\int_{\mathbb{R}^3} w_0^2(t,\xi) \bigl|\partial_{\beta_1} f(\xi)\bigr|^2 \, d\xi \Biggr]^{\frac{1}{2}} \Biggr\}$$

$$\times \Biggl[\int_{\mathbb{R}^3} \nu(\xi) w_\ell^2(t,\xi) \bigl|h(\xi)\bigr|^2 \, d\xi \Biggr]^{\frac{1}{2}}. \tag{2.11}$$

Proof. One can write $\Gamma(f,g)(\xi)$ as

$$\begin{aligned}
&\iint_{\mathbb{R}^3\times S^2} |\xi-\xi_*|^\gamma q_0(\theta) \mathbf{M}^{1/2}(\xi_*) f\bigl(\xi_*'\bigr) g\bigl(\xi'\bigr) \, d\omega \, d\xi_* \\
&\qquad - \iint_{\mathbb{R}^3\times S^2} |\xi-\xi_*|^\gamma q_0(\theta) \mathbf{M}^{1/2}(\xi_*) f(\xi_*) g(\xi) \, d\omega \, d\xi_* \\
&\quad = \iint_{\mathbb{R}^3\times S^2} |z|^\gamma q_0(\theta) \mathbf{M}^{1/2}(\xi+z) f(\xi+z_\perp) g(\xi+z_\parallel) \, d\omega \, dz \\
&\qquad - \iint_{\mathbb{R}^3\times S^2} |z|^\gamma q_0(\theta) \mathbf{M}^{1/2}(\xi+z) f(\xi+z) g(\xi) \, d\omega \, dz,
\end{aligned}$$

where as before, for given $\omega \in S^2$, $z_\parallel = z\cdot\omega\omega$ and $z_\perp = z - z_\parallel$. Then, $\partial_\beta \Gamma(f,g)(\xi)$ equals

$$\begin{aligned}
&\sum_{|\beta_0|+|\beta_1|+|\beta_2|\leqslant|\beta|} \iint_{\mathbb{R}^3\times S^2} |z|^\gamma q_0(\theta) \partial_{\beta_0} \mathbf{M}^{1/2}(\xi+z) \partial_{\beta_1} f(\xi+z_\perp) \partial_{\beta_2} g(\xi+z_\parallel) \, d\omega \, dz \\
&\quad - \sum_{|\beta_0|+|\beta_1|+|\beta_2|\leqslant|\beta|} \iint_{\mathbb{R}^3\times S^2} |z|^\gamma q_0(\theta) \partial_{\beta_0} \mathbf{M}^{1/2}(\xi+z) \partial_{\beta_1} f(\xi+z) \partial_{\beta_2} g(\xi) \, d\omega \, dz.
\end{aligned}$$

Using $\partial_{\beta_0} \mathbf{M}^{1/2}(\xi+z) \leqslant C\mathbf{M}^{q'/2}(\xi+z)$ for $0<q'<1$, it follows that

$$\left| \int_{\mathbb{R}^3} w_\ell^2(t,\xi) h(\xi) \partial_\beta \Gamma(f,g) \, d\xi \right| \leqslant C \sum_{|\beta_1|+|\beta_2|\leqslant|\beta|} (I_{1,\beta_1,\beta_2} + I_{2,\beta_1,\beta_2}) \tag{2.12}$$

with I_{1,β_1,β_2} and I_{2,β_1,β_2}, respectively, denoting

$$\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} |\xi-\xi_*|^\gamma q_0(\theta) \mathbf{M}^{q'/2}(\xi_*) \bigl|\partial_{\beta_1} f\bigl(\xi_*'\bigr)\bigr| \cdot \bigl|\partial_{\beta_2} g\bigl(\xi'\bigr)\bigr| w_\ell^2(t,\xi) \bigl|h(\xi)\bigr| \, d\omega \, d\xi \, d\xi_*$$

and

$$\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} |\xi-\xi_*|^\gamma q_0(\theta) \mathbf{M}^{q'/2}(\xi_*) \bigl|\partial_{\beta_1} f(\xi_*)\bigr| \cdot \bigl|\partial_{\beta_2} g(\xi)\bigr| w_\ell^2(t,\xi) \bigl|h(\xi)\bigr| \, d\omega \, d\xi \, d\xi_*.$$

For each (β_1, β_2) with $|\beta_1| + |\beta_2| \leqslant |\beta|$, from Hölder's inequality, I_{1,β_1,β_2} is bounded by

$$\left\{ \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} w_\ell^2(t,\xi)|\xi-\xi_*|^\gamma q_0(\theta)\mathbf{M}^{q'/2}(\xi_*)\left|\partial_{\beta_1} f\left(\xi'_*\right)\right|^2\left|\partial_{\beta_2} g\left(\xi'\right)\right|^2 d\omega\, d\xi\, d\xi_* \right\}^{\frac12}$$
$$\times \left\{ \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} w_\ell^2(t,\xi)|\xi-\xi_*|^\gamma q_0(\theta)\mathbf{M}^{q'/2}(\xi_*)\left|h(\xi)\right|^2 d\omega\, d\xi\, d\xi_* \right\}^{\frac12}.$$

Here, noticing

$$\iint_{\mathbb{R}^3\times S^2} |\xi-\xi_*|^\gamma q_0(\theta)\mathbf{M}^{q'/2}(\xi_*)\, d\omega\, d\xi_* \leqslant C\langle\xi\rangle^\gamma \leqslant C\nu(\xi),$$

one has

$$\begin{aligned} &\left\{ \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} w_\ell^2(t,\xi)|\xi-\xi_*|^\gamma q_0(\theta)\mathbf{M}^{q'/2}(\xi_*)\left|h(\xi)\right|^2 d\omega\, d\xi\, d\xi_* \right\}^{\frac12} \\ &\quad \leqslant C\left\{ \int_{\mathbb{R}^3} \nu(\xi) w_\ell^2(t,\xi)\left|h(\xi)\right|^2 d\xi \right\}^{\frac12}, \end{aligned}$$

and by using

$$\begin{aligned} &w_\ell^2(t,\xi)|\xi-\xi_*|^\gamma \mathbf{M}^{q'/2}(\xi_*) \\ &\quad \leqslant C\langle\xi\rangle^{\gamma+\ell} e^{\frac{2\lambda|\xi|}{(1+t)^\theta}} \\ &\quad \leqslant C\left\{\left\langle\xi'\right\rangle^{\gamma+\ell} + \left\langle\xi'_*\right\rangle^{\gamma+\ell}\right\} e^{\frac{2\lambda|\xi'|}{(1+t)^\theta}+\frac{2\lambda|\xi'_*|}{(1+t)^\theta}} \\ &\quad \leqslant C\left\{\nu\left(\xi'\right) w_\ell^2\left(t,\xi'\right) w_0^2\left(t,\xi'_*\right) + \nu\left(\xi'_*\right) w_\ell^2\left(t,\xi'_*\right) w_0^2\left(t,\xi'\right)\right\} \end{aligned}$$

which is due to $\gamma \geqslant 0$ and $\ell \geqslant 0$, one also has

$$\begin{aligned} &\left\{ \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} w_\ell^2(t,\xi)|\xi-\xi_*|^\gamma q_0(\theta)\mathbf{M}^{q'/2}(\xi_*)\left|\partial_{\beta_1} f\left(\xi'_*\right)\right|^2\left|\partial_{\beta_2} g\left(\xi'\right)\right|^2 d\omega\, d\xi\, d\xi_* \right\}^{\frac12} \\ &\quad \leqslant C\left\{ \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \nu\left(\xi'\right) w_\ell^2\left(t,\xi'\right) w_0^2\left(t,\xi'_*\right) q_0(\theta)\left|\partial_{\beta_1} f\left(\xi'_*\right)\right|^2\left|\partial_{\beta_2} g\left(\xi'\right)\right|^2 d\omega\, d\xi\, d\xi_* \right\}^{\frac12} \\ &\qquad + C\left\{ \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \nu\left(\xi'_*\right) w_\ell^2\left(t,\xi'_*\right) w_0^2\left(t,\xi'\right) q_0(\theta)\left|\partial_{\beta_1} f\left(\xi'_*\right)\right|^2\left|\partial_{\beta_2} g\left(\xi'\right)\right|^2 d\omega\, d\xi\, d\xi_* \right\}^{\frac12}, \end{aligned}$$

which is further bounded by

$$C\left\{ \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \nu(\xi) w_\ell^2(t,\xi) w_0^2(t,\xi_*) q_0(\theta) \left|\partial_{\beta_1} f(\xi_*)\right|^2 \left|\partial_{\beta_2} g(\xi)\right|^2 d\omega\, d\xi\, d\xi_* \right\}^{\frac{1}{2}}$$

$$+ C\left\{ \iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \nu(\xi_*) w_\ell^2(t,\xi_*) w_0^2(t,\xi) q_0(\theta) \left|\partial_{\beta_1} f(\xi_*)\right|^2 \left|\partial_{\beta_2} g(\xi)\right|^2 d\omega\, d\xi\, d\xi_* \right\}^{\frac{1}{2}}$$

$$\leqslant C\left\{ \left[\int\limits_{\mathbb{R}^3} \nu(\xi) w_\ell^2(t,\xi) \left|\partial_{\beta_2} g(\xi)\right|^2 d\xi \right]^{\frac{1}{2}} \left[\int\limits_{\mathbb{R}^3} w_0^2(t,\xi) \left|\partial_{\beta_1} f(\xi)\right|^2 d\xi \right]^{\frac{1}{2}} \right.$$

$$\left. + \left[\int\limits_{\mathbb{R}^3} \nu(\xi) w_\ell^2(t,\xi) \left|\partial_{\beta_1} f(\xi)\right|^2 d\xi \right]^{\frac{1}{2}} \left[\int\limits_{\mathbb{R}^3} w_0^2(t,\xi) \left|\partial_{\beta_2} g(\xi)\right|^2 d\xi \right]^{\frac{1}{2}} \right\}.$$

Here, we made change of variables $(\xi,\xi_*)\to(\xi',\xi_*')$ with the unit Jacobian. Therefore, by collecting the above estimates on I_{1,β_1,β_2}, $\sum_{|\beta_1|+|\beta_2|\leqslant|\beta|} I_{1,\beta_1,\beta_2}$ is bounded by the right-hand term of (2.11). In the simpler way, it is straightforward to verify that $\sum_{|\beta_1|+|\beta_2|\leqslant|\beta|} I_{2,\beta_1,\beta_2}$ is also bounded by the right-hand term of (2.11). Therefore, (2.11) follows from (2.12). This completes the proof of Lemma 2.3. □

3. Linearized time-decay

Consider the linearized system with a nonhomogeneous microscopic source:

$$\partial_t u + \xi\cdot\nabla_x u - \nabla_x\phi\cdot\xi \mathbf{M}^{1/2} = \mathbf{L}u + h, \qquad \mathbf{P}h = 0, \tag{3.1}$$

$$\phi(t,x) = -\frac{1}{4\pi|x|} *_x \int\limits_{\mathbb{R}^3} \mathbf{M}^{1/2} u(t,x,\xi)\, d\xi \to 0 \quad \text{as } |x|\to\infty. \tag{3.2}$$

Given initial data u_0, we formally define $e^{tB}u_0$ to be the solution to the linearized homogeneous system with $h\equiv 0$. For $1\leqslant q\leqslant 2$ and an integer m, set the index $\sigma_{q,m}$ of the time-decay rate in three dimensions by

$$\sigma_{q,m} = \frac{3}{2}\left(\frac{1}{q}-\frac{1}{2}\right) + \frac{m}{2}.$$

Theorem 3.1. *Assume $\int_{\mathbb{R}^3} a_0(x)\,dx = 0$, where $a_0(x) = a(0,x)$ is defined by u_0 in terms of $\mathbf{P}_0 u = a(t,x)\mathbf{M}^{1/2}$ at $t=0$. Then,*

$$\begin{aligned} &\left\|\partial_x^\alpha e^{t\mathbf{B}} u_0\right\| + \left\|\partial_x^\alpha \nabla_x \Delta_x^{-1} \mathbf{P}_0 e^{t\mathbf{B}} u_0\right\| \\ &\quad \leqslant C(1+t)^{-\sigma_{1,|\alpha|}}\left(\|u_0\|_{Z_1} + \left\|\partial_x^\alpha u_0\right\| + \left\||x| a_0\right\|_{L_x^1}\right), \end{aligned} \tag{3.3}$$

and in particular, when $\mathbf{P}_0 u_0 \equiv 0$,

$$\left\|\partial_x^\alpha e^{t\mathbf{B}} u_0\right\| + \left\|\partial_x^\alpha \nabla_x \Delta_x^{-1} \mathbf{P}_0 e^{t\mathbf{B}} u_0\right\| \leqslant C(1+t)^{-\sigma_{1,|\alpha|}}\left(\|u_0\|_{Z_1} + \left\|\partial_x^\alpha u_0\right\|\right). \tag{3.4}$$

Moreover, for $h = \{\mathbf{I}-\mathbf{P}\}h$,

$$\left\|\partial_x^\alpha \int_0^t e^{(t-s)\mathbf{B}}h(s)\,ds\right\|^2 + \left\|\partial_x^\alpha \nabla_x \Delta_x^{-1}\mathbf{P}_0 \int_0^t e^{(t-s)\mathbf{B}}h(s)\,ds\right\|^2$$
$$\leqslant C\int_0^t (1+t-s)^{-2\sigma_{1,|\alpha|}}\left(\left\|\nu^{-1/2}h(s)\right\|_{Z_1}^2 + \left\|\nu^{-1/2}\partial_x^\alpha h(s)\right\|^2\right)ds. \tag{3.5}$$

Proof. Both (3.4) and (3.5) were proved in [8, Theorem 2 on p. 303]. It suffices to prove (3.3). As in [8], recall that by letting $u = e^{tB}u_0$ be the solution to (3.1)–(3.2) when $h = 0$, there is a time–frequency functional $E(\hat{u}(t,k)) \sim \|\hat{u}(t,k)\|_{L^2_\xi}^2 + |\hat{a}(t,k)|^2/|k|^2$ such that

$$\frac{d}{dt}E\big(\hat{u}(t,k)\big) + \frac{c|k|^2}{1+|k|^2}E\big(\hat{u}(t,k)\big) \leqslant 0,$$

and hence $E(\hat{u}(t,k)) \leqslant e^{-\frac{c|k|^2}{1+|k|^2}t}E(\hat{u}(0,k))$. Setting $k^\alpha = k_1^{\alpha_1}k_2^{\alpha_2}k_3^{\alpha_3}$ and by noticing

$$\left\|\partial_x^\alpha e^{t\mathbf{B}}u_0\right\|^2 + \left\|\partial_x^\alpha \nabla_x\Delta_x^{-1}\mathbf{P}_0 e^{t\mathbf{B}}u_0\right\|^2 = \int_{\mathbb{R}^3_k}\left|k^{2\alpha}\right|\cdot\left\|\hat{u}(t,k)\right\|_{L^2_\xi}^2 dk + \int_{\mathbb{R}^3_k}\left|k^{2\alpha}\right|\cdot\frac{1}{|k|^2}\left|\hat{a}(t,k)\right|^2 dk$$
$$\leqslant C\int_{\mathbb{R}^3_k}\left|k^{2\alpha}\right|\left|E\big(\hat{u}(t,k)\big)\right|dk,$$

one has

$$\left\|\partial_x^\alpha e^{t\mathbf{B}}u_0\right\|^2 + \left\|\partial_x^\alpha \nabla_x\Delta_x^{-1}\mathbf{P}_0 e^{t\mathbf{B}}u_0\right\|^2$$
$$\leqslant C\int_{\mathbb{R}^3_k}\left|k^{2\alpha}\right|e^{-\frac{c|k|^2}{1+|k|^2}t}\left\|\hat{u}_0(k)\right\|_{L^2_\xi}^2 dk + C\int_{\mathbb{R}^3_k}\frac{|k^{2\alpha}|}{|k|^2}e^{-\frac{c|k|^2}{1+|k|^2}t}\left|\hat{a}_0(k)\right|^2 dk. \tag{3.6}$$

To prove (3.3), it now suffices to estimate the second term in the right-hand side of (3.6) over the low frequency domain $|k| \leqslant 1$, cf. [18]. In fact, due to the assumption $\int_{\mathbb{R}^3} a_0(x)\,dx = 0$, i.e. $\hat{a}_0(0) = 0$, one has

$$\left|\hat{a}_0(k)\right| = \left|\hat{a}_0(k) - \hat{a}_0(0)\right| \leqslant \int_{\mathbb{R}^3}|k\cdot x|\left|a_0(x)\right|dx \leqslant |k|\int_{\mathbb{R}^3}|x|\left|a_0(x)\right|dx.$$

Then,

$$\int_{|k|\leqslant 1}\frac{|k^{2\alpha}|}{|k|^2}e^{-\frac{c|k|^2}{1+|k|^2}t}\left|\hat{a}_0(k)\right|^2 dk \leqslant \int_{|k|\leqslant 1}\left|k^{2\alpha}\right|e^{-\frac{c}{2}|k|^2 t}dk\,\left\||x|a_0\right\|_{L^1_x}^2 \leqslant C(1+t)^{-2\sigma_{1,|\alpha|}}\left\||x|a_0\right\|_{L^1_x}^2.$$

This completes the proof of (3.3) and hence Theorem 3.1. □

4. *A priori* estimates

This section is devoted to deducing certain *a priori* estimates on the solutions to the VPB system. For this purpose, it is supposed that the Cauchy problem (1.4)–(1.6) of the VPB system admits a smooth solution $u(t,x,\xi)$ over $0 \leqslant t < T$ for $0 < T \leqslant \infty$. To make the presentation easy to read, we divide this section into four subsections. The first one is on the macro dissipation of the VPB system.

4.1. Macro dissipation

As in [8], by introducing

$$\Theta_{ij}(u) = \int_{\mathbb{R}^3} (\xi_i\xi_j - 1)\mathbf{M}^{1/2} u \, d\xi, \qquad \Lambda_i(u) = \frac{1}{10}\int_{\mathbb{R}^3} \big(|\xi|^2 - 5\big)\xi_i \mathbf{M}^{1/2} u \, d\xi,$$

one can derive from (1.4)–(1.5) a fluid-type system of equations

$$\begin{gathered} \partial_t a + \nabla_x \cdot b = 0, \\ \partial_t b + \nabla_x(a+2c) + \nabla_x \cdot \Theta\big(\{\mathbf{I}-\mathbf{P}\}u\big) - \nabla_x\phi = \nabla_x\phi a, \\ \partial_t c + \frac{1}{3}\nabla_x \cdot b + \frac{5}{3}\nabla_x \cdot \Lambda\big(\{\mathbf{I}-\mathbf{P}\}u\big) = \frac{1}{3}\nabla_x\phi \cdot b, \\ \Delta_x \phi = a, \end{gathered}$$

and

$$\begin{aligned} &\partial_t \Theta_{ij}\big(\{\mathbf{I}-\mathbf{P}\}u\big) + \partial_i b_j + \partial_j b_i - \frac{2}{3}\delta_{ij}\nabla_x \cdot b - \frac{10}{3}\delta_{ij}\nabla_x \cdot \Lambda\big(\{\mathbf{I}-\mathbf{P}\}u\big) \\ &\quad = \Theta_{ij}(r+g) - \frac{2}{3}\delta_{ij}\nabla_x\phi \cdot b, \end{aligned}$$

$$\partial_t \Lambda_i\big(\{\mathbf{I}-\mathbf{P}\}u\big) + \partial_i c = \Lambda_i(r+g)$$

with

$$r = -\xi \cdot \nabla_x\{\mathbf{I}-\mathbf{P}\}u + \mathbf{L}u, \qquad g = \frac{1}{2}\xi \cdot \nabla_x\phi u - \nabla_x\phi \cdot \nabla_\xi u + \Gamma(u,u).$$

Here, r is a linear term only related to the micro component $\{\mathbf{I}-\mathbf{P}\}u$ and g is a quadratic nonlinear term.

Our main result in this subsection can be stated as in the following

Lemma 4.1. *There is a temporal interactive functional $\mathcal{E}_N^{\mathrm{int}}(t)$ such that*

$$\big|\mathcal{E}_N^{\mathrm{int}}(t)\big| \leqslant C\bigg\{\|a\|^2 + \sum_{|\alpha|\leqslant N-1}\Big(\big\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u(t)\big\|^2 + \big\|\partial_x^\alpha\nabla_x(a,b,c)\big\|^2\Big)\bigg\} \tag{4.1}$$

and

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}_N^{\mathrm{int}}(t)+\kappa\left\{\|a\|^2+\sum_{|\alpha|\leqslant N-1}\left\|\partial_x^\alpha\nabla_x(a,b,c)\right\|^2\right\}\\
&\quad\leqslant C\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2+C\left\{\|u\|^2_{L^2_\xi(H^N_x)}+\|\nabla_x\phi\|^2_{H^N}\right\}\\
&\qquad\times\left\{\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2+\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial_x^\alpha(a,b,c)\right\|^2\right\}
\end{aligned}
\tag{4.2}
$$

hold for any $0\leqslant t<T$.

Proof. Basing on the analysis of the above macro fluid-type system, the desired estimates follow by the repeating the arguments employed in the proof of [8, Theorem 5.2] for the hard-sphere case and hence details are omitted. Here, we only point out the representation of $\mathcal{E}_N^{\mathrm{int}}(t)$ as

$$
\begin{aligned}
\mathcal{E}_N^{\mathrm{int}}(t)=&\sum_{|\alpha|\leqslant N-1}\int_{\mathbb{R}^3}\nabla_x\partial^\alpha c\cdot\Lambda\big(\partial^\alpha\{\mathbf{I}-\mathbf{P}\}u\big)\,dx\\
&+\sum_{|\alpha|\leqslant N-1}\sum_{ij=1}^{3}\int_{\mathbb{R}^3}\left(\partial_i\partial^\alpha b_j+\partial_j\partial^\alpha b_i-\frac{2}{3}\delta_{ij}\nabla_x\cdot\partial^\alpha b\right)\Theta_{ij}\big(\partial^\alpha\{\mathbf{I}-\mathbf{P}\}u\big)\,dx\\
&-\kappa\sum_{|\alpha|\leqslant N-1}\int_{\mathbb{R}^3}\partial^\alpha a\partial^\alpha\nabla_x\cdot b\,dx,
\end{aligned}
$$

for a constant $\kappa>0$ small enough. Here, for simplicity, we used ∂_j to denote ∂_{x_j} for each $j=1,2,3$. $\square$

4.2. Non-weighted energy estimates

Set

$$
\mathcal{E}_N(t)\sim\sum_{|\alpha|+|\beta|\leqslant N}\left\|\partial_\beta^\alpha u(t)\right\|^2+\left\|\nabla_x\phi(t)\right\|^2_{H^N},\tag{4.3}
$$

$$
\mathcal{D}_N(t)=\sum_{|\alpha|+|\beta|\leqslant N}\left\|\nu^{1/2}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u(t)\right\|^2+\sum_{|\alpha|\leqslant N-1}\left\|\partial^\alpha\nabla_x(a,b,c)(t)\right\|^2+\left\|a(t)\right\|^2,\tag{4.4}
$$

this subsection is concerned with the non-weighted energy estimates on the solutions of the VPB system based on the following *a priori* assumption (**A1**): There is $\delta>0$ small enough such that

$$
\sup_{0\leqslant t<T}\left\{\|\nabla_x\phi\|+(1+t)^{1+\theta}\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial^\alpha\nabla_x\phi(t)\right\|\right\}\leqslant\delta,\tag{4.5}
$$

where $\theta>0$ is a positive constant to be determined later.

For later use, let us write down the time evolution of $\{\mathbf{I}-\mathbf{P}\}u$:

$$
\begin{aligned}
&\partial_t\{\mathbf{I}-\mathbf{P}\}u+\xi\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}u+\nabla_x\phi\cdot\nabla_\xi\{\mathbf{I}-\mathbf{P}\}u-\frac{1}{2}\xi\cdot\nabla_x\phi\{\mathbf{I}-\mathbf{P}\}u+\nu(\xi)\{\mathbf{I}-\mathbf{P}\}u\\
&\quad=K\{\mathbf{I}-\mathbf{P}\}u+\Gamma(u,u)+\mathbf{P}\left\{\xi\cdot\nabla_x u+\nabla_x\phi\cdot\nabla_\xi u-\frac{1}{2}\xi\cdot\nabla_x\phi u\right\}\\
&\qquad-\left\{\xi\cdot\nabla_x+\nabla_x\phi\cdot\nabla_\xi-\frac{1}{2}\xi\cdot\nabla_x\phi\right\}\mathbf{P}u. \qquad (4.6)
\end{aligned}
$$

Our main result in this subsection is

Lemma 4.2. *Under the a priori assumption* (**A1**)*, there is $\mathcal{E}_N(t)$ satisfying* (4.3) *such that*

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}_N(t)+\kappa\mathcal{D}_N(t)\\
&\quad\leqslant C\left\{\mathcal{E}_N(t)^{1/2}+\mathcal{E}_N(t)\right\}\mathcal{D}_N(t)+\frac{C\delta}{(1+t)^{1+\theta}}\sum_{|\alpha|+|\beta|\leqslant N}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|\cdot\left|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2 n\,dx\,d\xi \qquad (4.7)
\end{aligned}
$$

holds for any $0\leqslant t<T$, where $\mathcal{D}_N(t)$ is given by (4.4)*.*

Proof. We proceed along the same line as in [9, Lemmas 4.4, 4.5 and 4.6]. The main difference now lies in the way to deal with the nonlinear term $\frac{1}{2}\xi\cdot\nabla_x\phi u$. For simplicity we shall only give the detailed estimates on that term. First of all, after multiplying (1.4) by u and integrating over $\mathbb{R}^3\times\mathbb{R}^3$, one has zero-order estimate:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|u\|^2+\|\nabla_x\phi\|^2-\int_{\mathbb{R}^3}|b|^2(a+2c)\,dx\right\}+\kappa\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)\left|\{\mathbf{I}-\mathbf{P}\}u\right|^2 dx\,d\xi\\
&\quad\leqslant C\|u\|_{L^2_\xi(H^2_x)}\left\{\left\|\nabla_x(a,b,c)\right\|^2+\left\|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}u\right\|^2\right\}\\
&\qquad+C\left\{\left\|(a,b,c,\nabla_x\phi)\right\|_{H^1}+\|\nabla_x\phi\|\cdot\|\nabla_x b\|\right\}\left\{\left\|\nabla_x(a,b,c)\right\|^2+\left\|\{\mathbf{I}-\mathbf{P}\}u\right\|^2_{L^2_\xi(H^1_x)}\right\}\\
&\qquad+\frac{C\delta}{(1+t)^{1+\theta}}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|\cdot\left|\{\mathbf{I}-\mathbf{P}\}u\right|^2 dx\,d\xi. \qquad (4.8)
\end{aligned}
$$

In fact, we need only to consider

$$
\begin{aligned}
\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\xi\cdot\nabla_x\phi u^2\,dx\,d\xi&=\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\xi\cdot\nabla_x\phi|\mathbf{P}u|^2\,dx\,d\xi+\iint_{\mathbb{R}^3\times\mathbb{R}^3}\xi\cdot\nabla_x\phi\mathbf{P}u\{\mathbf{I}-\mathbf{P}\}u\,dx\,d\xi\\
&\quad+\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\xi\cdot\nabla_x\phi\left|\{\mathbf{I}-\mathbf{P}\}u\right|^2 dx\,d\xi,
\end{aligned}
$$

where the first two terms on the right-hand side can be estimated as in [9, Lemma 4.4], and for the third term, by using the inequality $\|\nabla_x\phi\|_{L^\infty_x}\leqslant C\|\nabla_x^2\phi\|_{H^1_x}$ and the *a priori* assumption (**A1**), we have

$$
\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\xi\cdot\nabla_x\phi\left|\{\mathbf{I}-\mathbf{P}\}u\right|^2 dx\,d\xi\leqslant\frac{C\delta}{(1+t)^{1+\theta}}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|\cdot\left|\{\mathbf{I}-\mathbf{P}\}u\right|^2 dx\,d\xi.
$$

This proves (4.8).

Next, by applying ∂_x^α with $1 \leqslant |\alpha| \leqslant N$ to (1.4), multiplying it by $\partial_x^\alpha u$, integrating over $\mathbb{R}^3 \times \mathbb{R}^3$ and taking summation over $1 \leqslant |\alpha| \leqslant N$, one has the pure space-derivative estimate:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{1\leqslant|\alpha|\leqslant N}\left\{\left\|\partial_x^\alpha u\right\|^2+\left\|\partial_x^\alpha\nabla_x\phi\right\|^2\right\}+\kappa\sum_{1\leqslant|\alpha|\leqslant N}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)\left|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2dx\,d\xi\\
&\quad\leqslant C\left\{\|u\|_{L^2_\xi(H^N_x)}+\|\nabla_x\phi\|_{H^N}\right\}\sum_{1\leqslant|\alpha|\leqslant N}\left\{\left\|\nu^{1/2}\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2+\left\|\partial_x^\alpha(a,b,c)\right\|^2\right\}\\
&\qquad+C\left\|\nabla_x^2\phi\right\|_{H^{N-1}}\left\{\sum_{|\alpha|\leqslant N-1}\left\|\partial_x^\alpha\nabla_\xi\{\mathbf{I}-\mathbf{P}\}u\right\|\right\}\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial_x^\alpha u\right\|\\
&\qquad+\frac{C\delta}{(1+t)^{1+\theta}}\sum_{1\leqslant|\alpha|\leqslant N}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|\cdot\left|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2dx\,d\xi.
\end{aligned}\tag{4.9}$$

Again, let us only consider the estimate on

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3}\partial_x^\alpha\left(\frac{1}{2}\xi\cdot\nabla_x\phi u\right)\partial_x^\alpha u\,dx\,d\xi=\sum_{\beta\leqslant\alpha}C_{\alpha,\beta}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{1}{2}\xi\cdot\partial_x^{\alpha-\beta}\nabla_x\phi\partial_x^\beta u\partial_x^\alpha u\,dx\,d\xi.$$

By writing further $u=\mathbf{P}u+\{\mathbf{I}-\mathbf{P}\}u$, one has the estimate on the trouble term

$$\begin{aligned}
&\sum_{\beta\leqslant\alpha}C_{\alpha,\beta}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{1}{2}\xi\cdot\partial_x^{\alpha-\beta}\nabla_x\phi\partial_x^\beta\{\mathbf{I}-\mathbf{P}\}u\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\,dx\,d\xi\\
&\quad\leqslant C\int_{\mathbb{R}^3}|\xi|\cdot\left\|\nabla_x^2\phi\right\|_{H^{N-1}_x}\left\{\sum_{1\leqslant|\beta|\leqslant N}\left\|\partial_x^\beta\{\mathbf{I}-\mathbf{P}\}u\right\|_{L^2_x}\right\}\left\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|_{L^2_x}d\xi\\
&\quad\leqslant\frac{C\delta}{(1+t)^{1+\theta}}\sum_{1\leqslant|\alpha|\leqslant N}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|\cdot\left|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2dx\,d\xi
\end{aligned}$$

due to the *a priori* assumption (**A1**). This proved (4.9).

For the mixed space–velocity-derivative estimate, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{m=1}^N C_m\sum_{\substack{|\beta|=m\\|\alpha|+|\beta|\leqslant N}}\left\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2+\kappa\sum_{\substack{|\beta|\geqslant1\\|\alpha|+|\beta|\leqslant N}}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)\left|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2dx\,d\xi\\
&\quad\leqslant C\left\{\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2+\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial_x^\alpha(a,b,c)\right\|^2\right\}+C\left\{\|u\|_{H^N_{x,\xi}}+\|\nabla_x\phi\|_{H^N}\right\}\\
&\qquad\times\left\{\sum_{|\alpha|+|\beta|\leqslant N}\left\|\nu^{1/2}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2+\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial_x^\alpha(a,b,c)\right\|^2\right\}\\
&\qquad+\frac{C\delta}{(1+t)^{1+\theta}}\sum_{\substack{|\beta|\geqslant1\\|\alpha|+|\beta|\leqslant N}}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|\cdot\left|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2dx\,d\xi,
\end{aligned}\tag{4.10}$$

where $C_m > 0$ ($1 \leqslant m \leqslant N$) are properly chosen constants. In fact, it follows from applying ∂_β^α with $|\beta| = m$ and $|\alpha| + |\beta| \leqslant N$ to Eq. (4.6) of $\{\mathbf{I} - \mathbf{P}\}u$, multiplying it by $\partial_\beta^\alpha\{\mathbf{I} - \mathbf{P}\}u$, integrating over $\mathbb{R}^3 \times \mathbb{R}^3$, taking summation over $\{|\beta| = m, |\alpha| + |\beta| \leqslant N\}$ for each given $1 \leqslant m \leqslant N$ and then taking the proper linear combination of those $N - 1$ estimates with properly chosen constants $C_m > 0$ ($1 \leqslant m \leqslant N$). Let us only consider the estimate on

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \partial_\beta^\alpha \left(\frac{1}{2}\xi \cdot \nabla_x \phi \{\mathbf{I} - \mathbf{P}\}u\right) \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\}u \, dx \, d\xi$$

$$= \sum_{\alpha' \leqslant \alpha,\, \beta' \leqslant \beta} C_{\alpha'}^{\alpha} C_{\beta'}^{\beta} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{2} \partial_{\beta - \beta'} \xi \cdot \partial^{\alpha - \alpha'} \nabla_x \phi \partial_{\beta'}^{\alpha'} \{\mathbf{I} - \mathbf{P}\}u \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\}u \, dx \, d\xi.$$

From the Sobolev and Hölder inequalities, the term on the right-hand side of the above identity is bounded by

$$\left\{ \sum_{\alpha' \leqslant \alpha,\, \beta' < \beta} + \sum_{\alpha' \leqslant \alpha,\, \beta' = \beta} \right\} C_{\alpha'}^{\alpha} C_{\beta'}^{\beta} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \cdots$$

$$\leqslant C \left\| \nabla_x^2 \phi \right\|_{H^{N-1}} \sum_{\substack{|\beta| \leqslant m \\ |\alpha| + |\beta| \leqslant N}} \left\| \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\}u \right\|^2 + C \int_{\mathbb{R}^3} |\xi| \cdot \left\| \nabla_x^2 \phi \right\|_{H^{N-1}} \sum_{\substack{|\beta| = m \\ |\alpha| + |\beta| \leqslant N}} \left\| \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\}u \right\|_{L_x^2}^2 d\xi,$$

which by the *a priori* assumption (**A1**), is further bounded by

$$C \left\| \nabla_x^2 \phi \right\|_{H^{N-1}} \sum_{\substack{|\beta| \leqslant m \\ |\alpha| + |\beta| \leqslant N}} \left\| \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\}u \right\|^2 + \frac{C\delta}{(1+t)^{1+\theta}} \sum_{\substack{|\beta| = m \\ |\alpha| + |\beta| \leqslant N}} \iint_{\mathbb{R}^3} |\xi| \cdot \left| \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\}u \right|^2 dx \, d\xi.$$

This proved (4.10).

Finally, we take the linear combination of the above four estimates (4.8), (4.9), (4.10) and (4.2) as $\{[(4.8) + (4.9)] \times M_1 + (4.2)\} \times M_2 + (4.10)$ for constants $M_1 > 0$, $M_2 > 0$ large enough. Recall (4.1), (4.7) follows for a well-defined energy functional $\mathcal{E}_N(t)$ satisfying (4.3) and the energy dissipation rate $\mathcal{D}_N(t)$ given by (4.4). This completes the proof of Lemma 4.2. □

4.3. *Weighted energy estimates*

Recall (1.7) for the definition of the mixed time–velocity weight function $w_\ell(t, \xi)$ and set

$$\mathcal{E}_{N,\ell}(t) \sim \sum_{|\alpha| + |\beta| \leqslant N} \left\| w_\ell(t, \xi) \partial_\beta^\alpha u(t) \right\|^2 + \left\| \nabla_x \phi(t) \right\|_{H^N}^2, \tag{4.11}$$

$$\mathcal{D}_{N,\ell}(t) = \sum_{|\alpha| + |\beta| \leqslant N} \left\| \nu^{1/2} w_\ell(t, \xi) \partial_\beta^\alpha \{\mathbf{I} - \mathbf{P}\}u(t) \right\|^2 + \sum_{|\alpha| \leqslant N-1} \left\| \partial^\alpha \nabla_x (a, b, c)(t) \right\|^2 + \left\| a(t) \right\|^2, \tag{4.12}$$

we have

Lemma 4.3. *Let $\ell \geqslant 0$. Under the a priori assumption* (**A1**)*, there is $\mathcal{E}_{N,\ell}(t)$ satisfying* (4.11) *such that*

$$\frac{d}{dt}\mathcal{E}_{N,\ell}(t)+\kappa\mathcal{D}_{N,\ell}(t)\leqslant C\big\{\mathcal{E}_{N,\ell}(t)^{1/2}+\mathcal{E}_{N,\ell}(t)\big\}\mathcal{D}_{N,\ell}(t) \tag{4.13}$$

for any $0\leqslant t<T$, where $\mathcal{D}_{N,\ell}(t)$ is given by (4.12)*.*

Proof. For any given $\ell \geqslant 0$, to construct $\mathcal{E}_{N,\ell}(t)$, we perform the weighted energy estimates by the following three steps. To the end, $\langle\cdot,\cdot\rangle$ is used to denote the inner product over $L^2_{x,\xi}$ for brevity.

Step 1. Weighted estimate on zero-order of $\{\mathbf{I}-\mathbf{P}\}u$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|w_\ell(t,\xi)\{\mathbf{I}-\mathbf{P}\}u(t)\big\|^2+\kappa\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)w_\ell^2(t,\xi)\big|\{\mathbf{I}-\mathbf{P}\}u\big|^2\,dx\,d\xi\\
&\qquad+\frac{\kappa}{(1+t)^{1+\theta}}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|w_\ell^2(t,\xi)\big|\{\mathbf{I}-\mathbf{P}\}u\big|^2\,dx\,d\xi\\
&\quad\leqslant C\Big\{\big\|\langle\xi\rangle^{\frac{\ell-1}{2}}\{\mathbf{I}-\mathbf{P}\}u\big\|^2+\|\nabla_x u\|^2\Big\}+C\mathcal{E}_{N,\ell}(t)^{1/2}\mathcal{D}_{N,\ell}(t).
\end{aligned}\tag{4.14}$$

In fact, by multiplying (4.6) by $w_\ell^2(t,\xi)\{\mathbf{I}-\mathbf{P}\}u$ and taking integration over $\mathbb{R}^3\times\mathbb{R}^3$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|w_\ell(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\big\|^2+\big\langle\nu(\xi),w_\ell^2(t,\xi)\big|\{\mathbf{I}-\mathbf{P}\}u\big|^2\big\rangle\\
&\qquad+\Big\langle-\frac{1}{2}\frac{d}{dt}w_\ell^2(t,\xi),\big|\{\mathbf{I}-\mathbf{P}\}u\big|^2\Big\rangle+\Big\langle-\frac{1}{2}\xi\cdot\nabla_x\phi,w_\ell^2(t,\xi)\big|\{\mathbf{I}-\mathbf{P}\}u\big|^2\Big\rangle\\
&\quad=\big\langle K\{\mathbf{I}-\mathbf{P}\}u,w_\ell^2(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\big\rangle+\big\langle\Gamma(u,u),w_\ell^2(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\big\rangle\\
&\qquad+\big\langle-\nabla_x\phi\cdot\nabla_\xi\{\mathbf{I}-\mathbf{P}\}u,w_\ell^2(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\big\rangle\\
&\qquad+\Big\langle\mathbf{P}\Big\{\xi\cdot\nabla_x u+\nabla_x\phi\cdot\nabla_\xi u-\frac{1}{2}\xi\cdot\nabla_x\phi u\Big\},w_\ell^2(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\Big\rangle\\
&\qquad+\Big\langle-\Big\{\xi\cdot\nabla_x+\nabla_x\phi\cdot\nabla_\xi-\frac{1}{2}\xi\cdot\nabla_x\phi\Big\}\mathbf{P}u,w_\ell^2(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\Big\rangle.
\end{aligned}\tag{4.15}$$

The last two terms on the left-hand side of (4.15) are estimated as follows. Noticing

$$-\frac{1}{2}\frac{d}{dt}w_\ell^2(t,\xi)=\frac{\lambda\theta|\xi|}{(1+t)^{1+\theta}}w_\ell^2(t,\xi),$$

then it holds from the *a priori* assumption (**A1**) that

$$\begin{aligned}
&\Big\langle-\frac{1}{2}\frac{d}{dt}w_\ell^2(t,\xi),\big|\{\mathbf{I}-\mathbf{P}\}u\big|^2\Big\rangle+\Big\langle-\frac{1}{2}\xi\cdot\nabla_x\phi,w_\ell^2(t,\xi)\big|\{\mathbf{I}-\mathbf{P}\}u\big|^2\Big\rangle\\
&\quad\geqslant\Big\{\frac{\lambda\theta}{(1+t)^{1+\theta}}-C\|\nabla_x\phi\|_{L^\infty_x}\Big\}\big\langle|\xi|,w_\ell^2(t,\xi)\big|\{\mathbf{I}-\mathbf{P}\}u\big|^2\big\rangle
\end{aligned}$$

$$\geqslant \frac{\lambda\theta - C\delta}{(1+t)^{1+\theta}}\left\langle |\xi|, w_\ell^2(t,\xi)\left|\{\mathbf{I}-\mathbf{P}\}u\right|^2\right\rangle$$
$$\geqslant \frac{\lambda\theta}{2(1+t)^{1+\theta}}\left\langle |\xi|, w_\ell^2(t,\xi)\left|\{\mathbf{I}-\mathbf{P}\}u\right|^2\right\rangle, \tag{4.16}$$

where smallness of $\delta > 0$ in (**A1**) such that $C\delta \leqslant \frac{1}{2}\lambda\theta$ was used.

Now we turn to deal with the corresponding terms on the right-hand side of (4.15). First, from Lemma 2.2, we have

$$\left\langle K\{\mathbf{I}-\mathbf{P}\}u, w_\ell^2(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\right\rangle \leqslant \eta\left\| w_\ell(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\right\|^2 + C_\eta\left\|\langle\xi\rangle^{\frac{\ell-1}{2}}\{\mathbf{I}-\mathbf{P}\}u\right\|^2$$

for an arbitrary constant $\eta > 0$, and from Lemma 2.3,

$$\begin{aligned}
&\left\langle \Gamma(u,u), w_\ell^2(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\right\rangle \\
&\quad = \left\langle \Gamma(\mathbf{P}u,\mathbf{P}u) + \Gamma\big(\mathbf{P}u,\{\mathbf{I}-\mathbf{P}\}u\big) + \Gamma\big(\{\mathbf{I}-\mathbf{P}\}u,\mathbf{P}u\big)\right. \\
&\qquad \left. + \Gamma\big(\{\mathbf{I}-\mathbf{P}\}u,\{\mathbf{I}-\mathbf{P}\}u\big), w_\ell^2(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\right\rangle \leqslant C\mathcal{E}_{N,\ell}(t)^{1/2}\mathcal{D}_{N,\ell}(t),
\end{aligned}$$

where the Sobolev inequality $\|f\|_{L_x^\infty} \leqslant C\|\nabla_x f\|_{H_x^1}$ and the inequality $w_0(t,\xi) \leqslant w_\ell(t,\xi)$ due to $\ell \geqslant 0$ were used. Notice that

$$\nabla_\xi w_\ell^2(t,\xi) = \ell\langle\xi\rangle^{\ell-1}\frac{\xi}{|\xi|}e^{\frac{2\lambda|\xi|}{(1+t)^\theta}} + \langle\xi\rangle^\ell e^{\frac{2\lambda|\xi|}{(1+t)^\theta}}\frac{2\lambda}{(1+t)^\theta}\frac{\xi}{|\xi|},$$

which implies $|\nabla_\xi w_\ell^2(t,\xi)| \leqslant (2\lambda+\ell)w_\ell^2(t,\xi)$, and hence, by the *a priori* assumption (**A1**),

$$\begin{aligned}
\left\langle -\nabla_x\phi\cdot\nabla_\xi\{\mathbf{I}-\mathbf{P}\}u, w_\ell^2(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\right\rangle &= \left\langle \nabla_x\phi\cdot\frac{1}{2}\nabla_\xi w_\ell^2(t,\xi), \left|\{\mathbf{I}-\mathbf{P}\}u\right|^2\right\rangle \\
&\leqslant \frac{2\lambda+\ell}{2}\|\nabla_x\phi\|_{L_x^\infty}\left\|w_\ell(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\right\|^2 \\
&\leqslant C\delta\left\|w_\ell(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\right\|^2.
\end{aligned}$$

Here and hereafter, we skip the dependence of C on constants λ and ℓ. Finally, it holds that

$$\begin{aligned}
&\left\langle \mathbf{P}\left\{\xi\cdot\nabla_x u + \nabla_x\phi\cdot\nabla_\xi u - \frac{1}{2}\xi\cdot\nabla_x\phi u\right\}, w_\ell^2(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\right\rangle \\
&\quad \leqslant \eta\left\|\{\mathbf{I}-\mathbf{P}\}u\right\|^2 + \frac{C}{\eta}\|\nabla_x u\|^2 + C\|\nabla_x\phi\|_{H^2}\left\{\left\|\{\mathbf{I}-\mathbf{P}\}u\right\|^2 + \left\|\nabla_x(a,b,c)\right\|^2\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle -\left\{\xi\cdot\nabla_x + \nabla_x\phi\cdot\nabla_\xi - \frac{1}{2}\xi\cdot\nabla_x\phi\right\}\mathbf{P}u, w_\ell^2(t,\xi)\{\mathbf{I}-\mathbf{P}\}u\right\rangle \\
&\quad \leqslant \eta\left\|\{\mathbf{I}-\mathbf{P}\}u\right\|^2 + \frac{C}{\eta}\left\|\nabla_x(a,b,c)\right\|^2 + C\|\nabla_x\phi\|_{H^2}\left\{\left\|\{\mathbf{I}-\mathbf{P}\}u\right\|^2 + \left\|\nabla_x(a,b,c)\right\|^2\right\}.
\end{aligned}$$

Here $\eta > 0$ is an arbitrary constant. Therefore, by choosing a small constant $\eta > 0$ and also using smallness of $\delta > 0$, (4.14) follows from collecting all the above estimates into (4.15).

Step 2. Weighted estimate on pure space-derivative of u:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{1\leqslant|\alpha|\leqslant N}\left\|w_\ell(t,\xi)\partial_x^\alpha u(t)\right\|^2+\kappa\sum_{1\leqslant|\alpha|\leqslant N}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)w_\ell^2(t,\xi)\left|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2dx\,d\xi\\
&\quad+\frac{\kappa}{(1+t)^{1+\theta}}\sum_{1\leqslant|\alpha|\leqslant N}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|w_\ell^2(t,\xi)\left|\partial_x^\alpha u\right|^2dx\,d\xi\\
&\leqslant C\sum_{1\leqslant|\alpha|\leqslant N}\left\{\left\|\langle\xi\rangle^{\frac{1}{2}\max\{\ell-1,0\}}\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2+\left\|\partial_x^\alpha(a,b,c)\right\|^2\right\}+\|a\|^2\\
&\quad+C\delta\sum_{1\leqslant|\alpha|\leqslant N-1}\left\|w_\ell(t,\xi)\nabla_\xi\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2+C\mathcal{E}_{N,\ell}(t)^{1/2}\mathcal{D}_{N,\ell}(t).
\end{aligned}\tag{4.17}
$$

In fact, take $1\leqslant|\alpha|\leqslant N$, and by applying ∂_x^α to (1.4) with

$$
\mathbf{L}u=\mathbf{L}\{\mathbf{I}-\mathbf{P}\}u=-\nu\{\mathbf{I}-\mathbf{P}\}u+K\{\mathbf{I}-\mathbf{P}\}u,
$$

multiplying by $w_\ell^2(t,\xi)\partial_x^\alpha u$ and integrating over $\mathbb{R}^3\times\mathbb{R}^3$, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|w_\ell(t,\xi)\partial_x^\alpha u\right\|^2+\left\langle\nu(\xi)\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u,w_\ell^2(t,\xi)\partial_x^\alpha u\right\rangle\\
&\qquad+\left\langle-\frac{1}{2}\frac{d}{dt}w_\ell^2(t,\xi),\left|\partial_x^\alpha u\right|^2\right\rangle+\left\langle-\partial_x^\alpha\left(\frac{1}{2}\xi\cdot\nabla_x\phi u\right),w_\ell^2(t,\xi)\partial_x^\alpha u\right\rangle\\
&\quad=\left\langle K\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u,w_\ell^2(t,\xi)\partial_x^\alpha u\right\rangle+\left\langle\partial_x^\alpha\Gamma(u,u),w_\ell^2(t,\xi)\partial_x^\alpha u\right\rangle\\
&\qquad+\left\langle\partial_x^\alpha\nabla_x\phi\cdot\xi\mathbf{M}^{1/2},w_\ell^2(t,\xi)\partial_x^\alpha u\right\rangle+\left\langle-\partial_x^\alpha(\nabla_x\phi\cdot\nabla_\xi u),w_\ell^2(t,\xi)\partial_x^\alpha u\right\rangle.
\end{aligned}\tag{4.18}
$$

The left-hand terms of (4.18) are estimated as

$$
\begin{aligned}
&\left\langle\nu(\xi)\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u,w_\ell^2(t,\xi)\partial_x^\alpha u\right\rangle\\
&\quad=\left\langle\nu(\xi),w_\ell^2(t,\xi)\left|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2\right\rangle+\left\langle\nu(\xi)\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u,w_\ell^2(t,\xi)\partial_x^\alpha\mathbf{P}u\right\rangle\\
&\quad\geqslant\left\langle\nu(\xi),w_\ell^2(t,\xi)\left|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2\right\rangle-C\left\{\left\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2+\left\|\partial_x^\alpha(a,b,c)\right\|^2\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{1\leqslant|\alpha|\leqslant N}\left\{\left\langle-\frac{1}{2}\frac{d}{dt}w_\ell^2(t,\xi),\left|\partial_x^\alpha u\right|^2\right\rangle+\left\langle-\partial_x^\alpha\left(\frac{1}{2}\xi\cdot\nabla_x\phi u\right),w_\ell^2(t,\xi)\partial_x^\alpha u\right\rangle\right\}\\
&\quad=\frac{\lambda\theta}{(1+t)^{1+\theta}}\sum_{1\leqslant|\alpha|\leqslant N}\left\langle|\xi|,w_\ell^2(t,\xi)\left|\partial_x^\alpha u\right|^2\right\rangle\\
&\qquad-\sum_{1\leqslant|\alpha|\leqslant N}\sum_{|\alpha'|\leqslant|\alpha|}\left\langle-\frac{1}{2}\xi\cdot\partial_x^{\alpha-\alpha'}\nabla_x\phi\partial_x^{\alpha'}u,w_\ell^2(t,\xi)\partial_x^\alpha u\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \left\{ \frac{\lambda\theta}{(1+t)^{1+\theta}} - C \left\| \nabla_x^2 \phi \right\|_{H^{N-1}} \right\} \sum_{1\leqslant|\alpha|\leqslant N} \left\langle |\xi|, w_\ell^2(t,\xi) \left| \partial_x^\alpha u \right|^2 \right\rangle \\
&\geqslant \frac{\lambda\theta}{2(1+t)^{1+\theta}} \sum_{1\leqslant|\alpha|\leqslant N} \left\langle |\xi|, w_\ell^2(t,\xi) \left| \partial_x^\alpha u \right|^2 \right\rangle,
\end{aligned}
\tag{4.19}
$$

where as in Step 1, the *a priori* assumption (**A1**) was used.

The right-hand terms of (4.18) are estimated as follows. From Lemmas 2.2 and 2.3, respectively, one has

$$
\begin{aligned}
\left\langle K \partial_x^\alpha \{\mathbf{I}-\mathbf{P}\}u, w_\ell^2(t,\xi)\partial_x^\alpha u \right\rangle &= \left\langle K \partial_x^\alpha \{\mathbf{I}-\mathbf{P}\}u, w_\ell^2(t,\xi)\partial_x^\alpha \{\mathbf{I}-\mathbf{P}\}u \right\rangle + \left\langle K \partial_x^\alpha \{\mathbf{I}-\mathbf{P}\}u, w_\ell^2(t,\xi)\partial_x^\alpha \mathbf{P}u \right\rangle \\
&\leqslant \eta \left\| w_\ell(t,\xi)\partial_x^\alpha \{\mathbf{I}-\mathbf{P}\}u \right\|^2 + C_\eta \left\| \langle \xi \rangle^{\frac{\ell-1}{2}} \partial_x^\alpha \{\mathbf{I}-\mathbf{P}\}u \right\|^2 \\
&\quad + C\left\{ \left\| \partial_x^\alpha \{\mathbf{I}-\mathbf{P}\}u \right\|^2 + \left\| \partial_x^\alpha (a,b,c) \right\|^2 \right\}
\end{aligned}
$$

for an arbitrary constant $\eta > 0$, and

$$
\begin{aligned}
\left\langle \partial_x^\alpha \Gamma(u,u), w_\ell^2(t,\xi)\partial_x^\alpha u \right\rangle &= \sum_{\alpha_1+\alpha_2=\alpha} \left\langle \Gamma\left(\partial_x^{\alpha_1} u, \partial_x^{\alpha_2} u\right), w_\ell^2(t,\xi)\partial_x^\alpha u \right\rangle \\
&\leqslant C\mathcal{E}_{N,\ell}(t)^{1/2}\mathcal{D}_{N,\ell}(t).
\end{aligned}
$$

Moreover, for the rest two terms, one has

$$
\begin{aligned}
\left\langle \partial_x^\alpha \nabla_x \phi \cdot \xi \mathbf{M}^{1/2}, w_\ell^2(t,\xi)\partial_x^\alpha u \right\rangle &\leqslant \eta \left\| \partial_x^\alpha u \right\|^2 + C_\eta \left\| \partial_x^\alpha \nabla_x \phi \right\|^2 \\
&\leqslant C\eta \left\{ \left\| \partial_x^\alpha \{\mathbf{I}-\mathbf{P}\}u \right\|^2 + \left\| \partial_x^\alpha (a,b,c) \right\|^2 \right\} + C_\eta \sum_{|\alpha'|=|\alpha|-1} \left\| \partial_x^{\alpha'} a \right\|^2
\end{aligned}
$$

for any $\eta > 0$, and

$$
\begin{aligned}
&\left\langle -\partial_x^\alpha (\nabla_x \phi \cdot \nabla_\xi u), w_\ell^2(t,\xi)\partial_x^\alpha u \right\rangle \\
&\quad = \left\langle \nabla_x \phi \left| \partial_x^\alpha u \right|^2, \frac{1}{2}\nabla_\xi w_\ell^2(t,\xi) \right\rangle + \sum_{\alpha'<\alpha} C_{\alpha'}^{\alpha} \left\langle -\partial_x^{\alpha-\alpha'} \nabla_x \phi \cdot \nabla_\xi \partial_x^{\alpha'} u, w_\ell^2(t,\xi)\partial_x^\alpha u \right\rangle \\
&\quad \leqslant \frac{2\lambda+\gamma}{2} \|\nabla_x \phi\|_{L_x^\infty} \left\| w_\ell(t,\xi)\partial_x^\alpha u \right\|^2 \\
&\qquad + C \left\| \nabla_x^2 \phi \right\|_{H^{N-1}} \left\{ \left\| w_\ell(t,\xi)\partial_x^\alpha u \right\|^2 + \sum_{1\leqslant|\alpha|\leqslant N-1} \left\| w_\ell(t,\xi)\nabla_\xi \partial_x^\alpha u \right\|^2 \right\} \\
&\quad \leqslant C\delta \left\{ \left\| w_\ell(t,\xi)\partial_x^\alpha \{\mathbf{I}-\mathbf{P}\}u \right\|^2 + \sum_{1\leqslant|\alpha|\leqslant N} \left\| \partial_x^\alpha (a,b,c) \right\|^2 \right\} \\
&\qquad + C\delta \sum_{1\leqslant|\alpha|\leqslant N-1} \left\| w_\ell(t,\xi)\nabla_\xi \partial_x^\alpha \{\mathbf{I}-\mathbf{P}\}u \right\|^2.
\end{aligned}
$$

Therefore, by choosing a small constant $\eta > 0$ and also using smallness of $\delta > 0$, (4.17) follows from plugging all the above estimates into (4.18) and then taking summation over $1 \leqslant |\alpha| \leqslant N$.

Step 3. For later use, set

$$\widetilde{\mathcal{D}}_{N,\ell}(t)=\sum_{|\alpha|+|\beta|\leqslant N}\left\|\langle\xi\rangle^{\frac{\ell}{2}}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u(t)\right\|^2+\sum_{|\alpha|\leqslant N-1}\left\|\partial^\alpha\nabla_x(a,b,c)(t)\right\|^2+\left\|a(t)\right\|^2, \tag{4.20}$$

one has the weighted estimate on mixed space–velocity-derivative of $\{\mathbf{I}-\mathbf{P}\}u$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{m=1}^N C_m\sum_{\substack{|\beta|=m\\|\alpha|+|\beta|\leqslant N}}\left\|w_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\|^2\\
&\quad+\kappa\sum_{\substack{|\beta|\geqslant 1\\|\alpha|+|\beta|\leqslant N}}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)w^2_\ell(t,\xi)\left|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right|^2dx\,d\xi\\
&\quad+\frac{\kappa}{(1+t)^{1+\theta}}\sum_{\substack{|\beta|\geqslant 1\\|\alpha|+|\beta|\leqslant N}}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|w^2_\ell(t,\xi)\left|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right|^2dx\,d\xi\\
&\leqslant C\widetilde{\mathcal{D}}_{N,\max\{\ell-1,0\}}(t)+C\sum_{|\alpha|\leqslant N}\left\|w_\ell(t,\xi)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2+C\mathcal{E}_{N,\ell}(t)^{1/2}\mathcal{D}_{N,\ell}(t).
\end{aligned} \tag{4.21}$$

In fact, let $1\leqslant m\leqslant N$. By applying ∂^α_β with $|\beta|=m$ and $|\alpha|+|\beta|\leqslant N$ to Eq. (4.6) of $\{\mathbf{I}-\mathbf{P}\}u$, multiplying it by $w^2_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u$ and integrating over $\mathbb{R}^3\times\mathbb{R}^3$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|w_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\|^2+\left\langle\nu(\xi),w^2_\ell(t,\xi)\left|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right|^2\right\rangle\\
&\quad+\left\langle-\frac{1}{2}\frac{d}{dt}w^2_\ell(t,\xi),\left|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right|^2\right\rangle+\left\langle\partial^\alpha_\beta\left(-\frac{1}{2}\xi\cdot\nabla_x\phi\{\mathbf{I}-\mathbf{P}\}u\right),w^2_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&=\left\langle\partial^\alpha_\beta K\{\mathbf{I}-\mathbf{P}\}u,w^2_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\rangle+\left\langle\partial^\alpha_\beta\Gamma(u,u),w^2_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&\quad+\left\langle-\partial^\alpha_\beta\left(\nabla_x\phi\cdot\nabla_\xi\{\mathbf{I}-\mathbf{P}\}u\right),w^2_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&\quad+\left\langle-\left[\!\left[\partial^\alpha_\beta,\xi\cdot\nabla_x+\nu(\xi)\right]\!\right]\{\mathbf{I}-\mathbf{P}\}u,w^2_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&\quad+\left\langle\partial^\alpha_\beta\mathbf{P}\left\{\xi\cdot\nabla_x u+\nabla_x\phi\cdot\nabla_\xi u-\frac{1}{2}\xi\cdot\nabla_x\phi u\right\},w^2_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&\quad+\left\langle-\partial^\alpha_\beta\left\{\xi\cdot\nabla_x+\nabla_x\phi\cdot\nabla_\xi-\frac{1}{2}\xi\cdot\nabla_x\phi\right\}\mathbf{P}u,w^2_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\rangle,
\end{aligned} \tag{4.22}$$

where $[\![\cdot,\cdot]\!]$ denotes the usual commutator. Similarly as in Step 2, by noticing the identity

$$\begin{aligned}
&\partial^\alpha_\beta\left(-\frac{1}{2}\xi\cdot\nabla_x\phi\{\mathbf{I}-\mathbf{P}\}u\right)\\
&=-\frac{1}{2}\sum_{\alpha'\leqslant\alpha}C^\alpha_{\alpha'}\xi\cdot\nabla_x\partial^{\alpha-\alpha'}_x\phi\partial^{\alpha'}_\beta\{\mathbf{I}-\mathbf{P}\}u-\frac{1}{2}\sum_{\alpha'\leqslant\alpha}\sum_{\beta'<\beta}C^\alpha_{\alpha'}C^\beta_{\beta'}\partial_{\beta-\beta'}\xi\cdot\nabla_x\partial^{\alpha-\alpha'}_x\phi\partial^{\alpha'}_{\beta'}\{\mathbf{I}-\mathbf{P}\}u
\end{aligned}$$

and using the *a priori* assumption (**A1**), the left-hand terms of (4.22) are estimated as

$$
\begin{aligned}
&\sum_{\substack{|\beta|=m\\|\alpha|+|\beta|\leqslant N}}\left\{\left\langle -\frac{1}{2}\frac{d}{dt}w_\ell^2(t,\xi), \left|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2\right\rangle\right.\\
&\qquad\left.+\left\langle \partial_\beta^\alpha\left(-\frac{1}{2}\xi\cdot\nabla_x\phi\{\mathbf{I}-\mathbf{P}\}u\right), w_\ell^2(t,\xi)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\rangle\right\}\\
&\quad\geqslant \frac{\lambda\theta}{2(1+t)^{1+\theta}}\sum_{\substack{|\beta|=m\\|\alpha|+|\beta|\leqslant N}}\left\langle |\xi|, w_\ell^2(t,\xi)\left|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2\right\rangle\\
&\qquad - C\delta\sum_{\substack{|\beta|\leqslant m\\|\alpha|+|\beta|\leqslant N}}\left\|w_\ell(t,\xi)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2. \qquad (4.23)
\end{aligned}
$$

The right-hand terms of (4.22) are estimated as follows. Lemma 2.2 implies

$$
\begin{aligned}
&\left\langle \partial_\beta^\alpha K\{\mathbf{I}-\mathbf{P}\}u, w_\ell^2(t,\xi)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&\quad=\left\langle \partial_\beta K\partial^\alpha\{\mathbf{I}-\mathbf{P}\}u, w_\ell^2(t,\xi)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&\quad\leqslant \eta\iint_{\mathbb{R}^3\times\mathbb{R}^3} w_\ell^2(t,\xi)\left\{\left|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2+\sum_{|\beta'|\leqslant|\beta|}\left|\partial_{\beta'}^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2\right\}dx\,d\xi\\
&\qquad + C_{|\beta|,\ell,\eta}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\langle\xi\rangle^{\ell-1}\left\{\left|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2+\sum_{|\beta'|\leqslant|\beta|}\left|\partial_{\beta'}^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2\right\}dx\,d\xi
\end{aligned}
$$

for any $\eta>0$, and Lemma 2.3 together with Sobolev inequalities implies

$$
\begin{aligned}
\left\langle \partial_\beta^\alpha\Gamma(u,u), w_\ell^2(t,\xi)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\rangle &= \sum_{\alpha_1+\alpha_2=\alpha}\left\langle \partial_\beta\Gamma\left(\partial^{\alpha_1}u,\partial^{\alpha_2}u\right), w_\ell^2(t,\xi)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&\leqslant C\mathcal{E}_{N,\ell}(t)^{1/2}\mathcal{D}_{N,\ell}(t).
\end{aligned}
$$

Moreover, as in Step 2,

$$
\begin{aligned}
&\left\langle -\partial_\beta^\alpha\left(\nabla_x\phi\cdot\nabla_\xi\{\mathbf{I}-\mathbf{P}\}u\right), w_\ell^2(t,\xi)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&\quad=\left\langle \nabla_x\phi\left|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right|^2, \frac{1}{2}\nabla_\xi w_\ell^2(t,\xi)\right\rangle\\
&\qquad+\sum_{\alpha'<\alpha}C_{\alpha'}^{\alpha}\left\langle -\partial_x^{\alpha-\alpha'}\nabla_x\phi\cdot\nabla_\xi\partial_\beta^{\alpha'}\{\mathbf{I}-\mathbf{P}\}u, w_\ell^2(t,\xi)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&\quad\leqslant\frac{2\lambda+\ell}{2}\|\nabla_x\phi\|_{L_x^\infty}\left\|w_\ell(t,\xi)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2 + C\left\|\nabla_x^2\phi\right\|_{H^{N-1}}\sum_{\substack{|\beta|\geqslant 1\\|\alpha|+|\beta|\leqslant N}}\left\|w_\ell(t,\xi)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2\\
&\quad\leqslant C\delta\sum_{\substack{|\beta|\geqslant 1\\|\alpha|+|\beta|\leqslant N}}\left\|w_\ell(t,\xi)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}u\right\|^2.
\end{aligned}
$$

Finally, it is straightforward to obtain

$$\begin{aligned}
&\left\langle -\left[\!\left[\partial^\alpha_\beta, \xi\cdot\nabla_x+\nu(\xi)\right]\!\right]\{\mathbf{I}-\mathbf{P}\}u, w^2_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&\quad\leqslant \eta\left\|w_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\|^2+C_\eta\sum_{\substack{|\beta|\leqslant m-1\\ |\alpha|+|\beta|\leqslant N}}\left\|w_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle \partial^\alpha_\beta\mathbf{P}\left\{\xi\cdot\nabla_x u+\nabla_x\phi\cdot\nabla_\xi u-\frac{1}{2}\xi\cdot\nabla_x\phi u\right\}, w^2_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&\qquad+\left\langle -\partial^\alpha_\beta\left\{\xi\cdot\nabla_x+\nabla_x\phi\cdot\nabla_\xi-\frac{1}{2}\xi\cdot\nabla_x\phi\right\}\mathbf{P}u, w^2_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\rangle\\
&\quad\leqslant \eta\left\|w_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\|^2+C_\eta\left\{\left\|\nabla_x\partial^\alpha_x u\right\|^2+\left\|\nabla_x\partial^\alpha_x(a,b,c)\right\|^2\right\}\\
&\qquad+C\|\nabla_x\phi\|_{H^{N-1}}\sum_{1\leqslant|\alpha|\leqslant N}\left\{\left\|\partial^\alpha_x u\right\|^2+\left\|\partial^\alpha_x(a,b,c)\right\|^2\right\}\\
&\quad\leqslant \eta\left\|w_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right\|^2+C_\eta\sum_{1\leqslant|\alpha|\leqslant N}\left\{\left\|\partial^\alpha_x\{\mathbf{I}-\mathbf{P}\}u\right\|^2+\left\|\partial^\alpha_x(a,b,c)\right\|^2\right\}
\end{aligned}$$

for any $\eta>0$. Therefore, one can choose a small constant $\eta>0$ and use smallness of $\delta>0$ and (4.20) for the definition of $\widetilde{\mathcal{D}}_{N,\ell}(t)$, so that (4.21) follows by plugging all the estimates above into (4.22), taking summation over $\{|\beta|=m, |\alpha|+|\beta|\leqslant N\}$ for each given $1\leqslant m\leqslant N$ and then taking the proper linear combination of those $N-1$ estimates with properly chosen constants $C_m>0$ $(1\leqslant m\leqslant N)$.

We are now in a position to prove (4.13) by induction on $\ell\geqslant 0$. When $0\leqslant\ell\leqslant 1$, (4.13) with properly defined $\mathcal{E}_{N,\ell}(t)$ follows from the linear combination

$$\left\{\left[(4.14)+(4.17)\right]\times M_1+(4.21)\right\}+(4.7)\times M_2$$

for properly chosen constants $M_2\gg M_1\gg 1$ large enough. In fact, in the linear combination $[(4.14)+(4.17)]\times M_1+(4.21)$, one can first take $M_1>0$ large enough in order to absorb the right-hand second term of (4.21), and in the mean time the right-hand third term of (4.17) is absorbed by the dissipation terms in (4.21) since $\delta>0$ is small enough. In the further linear combination with (4.7), one can take $M_2>0$ large enough to absorb all the right-hand dissipation terms without velocity weighted functions due to $0\leqslant\ell\leqslant 1$, and meanwhile, again thanks to smallness of $\delta>0$, the right-hand second term of (4.7) is also absorbed by the dissipation terms obtained in the previous step. This proves (4.13) in the case $0\leqslant\ell\leqslant 1$. Next, assume that (4.13) is true for $\ell-1$ with $\ell\geqslant 1$, i.e.

$$\frac{d}{dt}\mathcal{E}_{N,\ell-1}(t)+\kappa\mathcal{D}_{N,\ell-1}(t)\leqslant C\left\{\mathcal{E}_{N,\ell-1}(t)^{1/2}+\mathcal{E}_{N,\ell-1}(t)\right\}\mathcal{D}_{N,\ell-1}(t). \tag{4.24}$$

Then, in the completely same way as before, (4.13) with properly defined $\mathcal{E}_{N,\ell}(t)$ follows from the linear combination

$$\left\{\left[(4.14)+(4.17)\right]\times M_1+(4.21)\right\}+(4.24)\times M_2$$

for properly chosen constants $M_2\gg M_1\gg 1$ large enough. Here, the fact that $\widetilde{\mathcal{D}}_{N,\max\{\ell-1,0\}}(t)\leqslant C\mathcal{D}_{N,\ell-1}(t)$ with $\ell\geqslant 1$ was used. Hence, by induction on ℓ, (4.13) holds true for any given $\ell\geqslant 0$. This completes the proof of Lemma 4.3. □

4.4. High-order energy estimates

Set

$$\mathcal{E}^{\mathrm{h}}_{N,\ell}(t) \sim \sum_{|\alpha|+|\beta|\leqslant N} \left\| w_\ell(t,\xi)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u(t)\right\|^2 + \left\|\nabla_x^2\phi(t)\right\|^2_{H^{N-1}} + \sum_{|\alpha|\leqslant N-1}\left\|\partial^\alpha\nabla_x(a,b,c)(t)\right\|^2, \tag{4.25}$$

we now deduce the high-order energy estimates based on another *a priori* assumption (**A2**): There is $\delta > 0$ small enough such that

$$\sup_{0\leqslant t<T} \mathcal{E}_{N,\ell}(t) \leqslant \delta. \tag{4.26}$$

Our main result in this subsection can be stated in the following

Lemma 4.4. *Let $\ell \geqslant 0$. Under the a priori assumptions* (**A1**) *and* (**A2**)*, there is $\mathcal{E}^{\mathrm{h}}_{N,\ell}(t)$ satisfying* (4.25) *such that*

$$\frac{d}{dt}\mathcal{E}^{\mathrm{h}}_{N,\ell}(t) + \kappa\mathcal{D}_{N,\ell}(t) \leqslant C\left\|\nabla_x(a,b,c)\right\|^2 \tag{4.27}$$

holds for any $0 \leqslant t < T$, where $\mathcal{D}_{N,\ell}(t)$ is given by (4.12)*.*

Proof. We proceed along the same line as in the proof of Lemma 4.3. First of all, similar to that of Lemma 4.2, we claim that under the *a priori* assumption (**A1**), there is $\mathcal{E}^{\mathrm{h}}_N(t)$ with

$$\mathcal{E}^{\mathrm{h}}_N(t) \sim \left\|\{\mathbf{I}-\mathbf{P}\}u(t)\right\|^2 + \sum_{1\leqslant|\alpha|\leqslant N}\left\{\left\|\partial^\alpha u(t)\right\|^2 + \left\|\partial^\alpha\nabla_x\phi(t)\right|^2\right\} + \sum_{\substack{|\beta|\geqslant 1\\ |\alpha|+|\beta|\leqslant N}}\left\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u(t)\right\|^2 \tag{4.28}$$

such that

$$\frac{d}{dt}\mathcal{E}^{\mathrm{h}}_N(t) + \kappa\mathcal{D}_N(t) \leqslant C\left\|\nabla_x(a,b,c)\right\|^2 + C\left\{\mathcal{E}_N(t)^{1/2} + \mathcal{E}_N(t)\right\}\mathcal{D}_N(t) + \frac{C\delta}{(1+t)^{1+\theta}}\sum_{|\alpha|+|\beta|\leqslant N}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|\cdot\left|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right|^2 dx\,d\xi, \tag{4.29}$$

where $\mathcal{D}_N(t)$ is given by (4.4). In fact, it suffices to replace (4.8) by the following zero-order estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\{\mathbf{I}-\mathbf{P}\}u(t)\right\|^2 + \kappa\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)\left|\{\mathbf{I}-\mathbf{P}\}u\right|^2 dx\,d\xi \\ \leqslant C\left\|\nabla_x(a,b,c)\right\|^2 + C\left\{\mathcal{E}_N(t)^{1/2} + \mathcal{E}_N(t)\right\}\mathcal{D}_N(t) \\ + \frac{C\delta}{(1+t)^{1+\theta}}\sum_{|\alpha|+|\beta|\leqslant N}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|\cdot\left|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}u\right|^2 dx\,d\xi.\end{aligned} \tag{4.30}$$

Then, by exploiting the same argument used in the proof of Lemma 4.2, (4.29) with a properly defined energy functional $\mathcal{E}_N^{\mathrm{h}}(t)$ satisfying (4.28) follows from the proper linear combination of (4.30), (4.9), (4.10) and (4.2). Here, by (4.1), we used the fact that

$$\left|\mathcal{E}_N^{\mathrm{int}}(t)\right| \leqslant C\left\{\left\|\{\mathbf{I}-\mathbf{P}\}u(t)\right\|^2 + \sum_{1\leqslant|\alpha|\leqslant N}\left(\left\|\partial_x^\alpha u(t)\right\|^2 + \left\|\partial_x^\alpha \nabla_x\phi(t)\right\|^2\right)\right\}.$$

To prove (4.30), by multiplying (4.6) by $\{\mathbf{I}-\mathbf{P}\}u$ and integrating over $\mathbb{R}^3\times\mathbb{R}^3$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\{\mathbf{I}-\mathbf{P}\}u(t)\right\|^2 + \left\langle \mathbf{L}\{\mathbf{I}-\mathbf{P}\}u, \{\mathbf{I}-\mathbf{P}\}u\right\rangle \\
&\quad = \left\langle \Gamma(u,u), \{\mathbf{I}-\mathbf{P}\}u\right\rangle + \left\langle \frac{1}{2}\xi\cdot\nabla_x\phi\{\mathbf{I}-\mathbf{P}\}u, \{\mathbf{I}-\mathbf{P}\}u\right\rangle \\
&\qquad + \left\langle -\left\{\xi\cdot\nabla_x + \nabla_x\phi\cdot\nabla_\xi - \frac{1}{2}\xi\cdot\nabla_x\phi\right\}\mathbf{P}u, \{\mathbf{I}-\mathbf{P}\}u\right\rangle.
\end{aligned}$$

Here, the right-hand third term is bounded by

$$\eta\left\|\{\mathbf{I}-\mathbf{P}\}u\right\|^2 + C_\eta\left(1+\|\nabla_x\phi\|_{H^1}^2\right)\left\|\nabla_x(a,b,c)\right\|^2$$

for $\eta>0$ chosen properly small. Other terms can be estimated as in obtaining (4.8). This hence proves (4.30).

Next, one can repeat the proof of (4.13) in Lemma 4.3 by replacing (4.7) by (4.30) so that for any given $\ell\geqslant 0$, there is $\mathcal{E}_{N,\ell}^{\mathrm{h}}(t)$ satisfying (4.25) such that

$$\frac{d}{dt}\mathcal{E}_{N,\ell}^{\mathrm{h}}(t) + \kappa\mathcal{D}_{N,\ell}(t) \leqslant C\left\|\nabla_x(a,b,c)\right\|^2 + C\left\{\mathcal{E}_{N,\ell}(t)^{1/2} + \mathcal{E}_{N,\ell}(t)\right\}\mathcal{D}_{N,\ell}(t),$$

which further implies (4.27) by the assumption (**A2**) and smallness of $\delta>0$. This completes the proof of Lemma 4.4. □

5. Global existence

Recall (4.11) and (4.25) and for $t\geqslant 0$, define

$$X_{N,\ell}(t) = \sup_{0\leqslant s\leqslant t}(1+s)^{\frac{3}{2}}\mathcal{E}_{N,\ell}(s) + \sup_{0\leqslant s\leqslant t}(1+s)^{\frac{5}{2}}\mathcal{E}_{N,\ell}^{\mathrm{h}}(s). \tag{5.1}$$

Our main result in this section is

Lemma 5.1. *Let $N\geqslant 4$, and let $\ell\geqslant 2$, $\lambda>0$ and $\theta>0$ in $w_\ell(t,\xi)$ given by* (1.7)*. Assume $\int_{\mathbb{R}^3} a_0(x)\,dx = 0$. Then, under the a priori assumptions* (**A1**) *and* (**A2**)*, any smooth solution $u(t,x,\xi)$ to the Cauchy problem* (1.4)–(1.6) *of the VPB system over $0\leqslant t<T$ with $0<T\leqslant\infty$ satisfies*

$$X_{N,\ell}(t) \leqslant C\left\{\mathcal{E}_{N,\ell}(0) + \|u_0\|_{Z_1}^2 + \left\||x|a_0\right\|_{L_x^1}^2\right\} + CX_{N,\ell}(t)^2 \tag{5.2}$$

for $0\leqslant t<T$ and some positive constants independent of T.

Proof. Take $N \geqslant 4$, $\ell \geqslant 2$, $\lambda > 0$ and $\theta > 0$. Let (**A1**), (**A2**) hold with $\delta > 0$ small enough. It follows from Lemma 4.3 that

$$\frac{d}{dt}\mathcal{E}_{N,\ell}(t) + \kappa \mathcal{D}_{N,\ell}(t) \leqslant 0. \tag{5.3}$$

By comparing (4.11) with (4.12), there is $\kappa > 0$ such that

$$\mathcal{D}_{N,\ell}(t) + \big\|(b, c, \nabla_x \phi)\big\|^2 \geqslant \kappa \mathcal{E}_{N,\ell}(t).$$

Recall (2.2) for the definitions of $\mathbf{P}_0$ and $\mathbf{P}_1$. Applying the above inequality together with the observation $\|(b, c, \nabla_x \phi)\|^2 \leqslant C(\|\mathbf{P}_1 u(t)\|^2 + \|\nabla_x \Delta_x^{-1} \mathbf{P}_0 u(t)\|^2)$ to (5.3), one has

$$\frac{d}{dt}\mathcal{E}_{N,\ell}(t) + \kappa \mathcal{E}_{N,\ell}(t) \leqslant C\big(\big\|\mathbf{P}_1 u(t)\big\|^2 + \big\|\nabla_x \Delta_x^{-1}\mathbf{P}_0 u(t)\big\|^2\big). \tag{5.4}$$

The right-hand terms of (5.4) are estimated as follows. Recall that the solution u to the Cauchy problem (1.4)–(1.6) of the nonlinear VPB system can be written as the mild form

$$u(t) = e^{t\mathbf{B}} u_0 + \int_0^t e^{(t-s)\mathbf{B}} g(s)\, ds \tag{5.5}$$

with $g = \frac{1}{2}\xi \cdot \nabla_x \phi u - \nabla_x \phi \cdot \nabla_\xi u + \Gamma(u, u)$. For later use, write g as $g = g_1 + g_2$ with $g_1 = \Gamma(u, u)$ and $g_2 = \frac{1}{2}\xi \cdot \nabla_x \phi u - \nabla_x \phi \cdot \nabla_\xi u$. Observe that $\mathbf{P} g_1 = 0$ and $\mathbf{P}_0 g_2 = 0$. Then, one can rewrite g as $g = \{\mathbf{I} - \mathbf{P}\} g_1 + \{\mathbf{I} - \mathbf{P}\} g_2 + \mathbf{P}_1 g_2$. With the help of this representation for g, applying Theorem 3.1 to (5.5) gives

$$\begin{aligned} \big\|\mathbf{P}_1 u(t)\big\|^2 + \big\|\nabla_x \Delta_x^{-1} \mathbf{P}_0 u(t)\big\|^2 \leqslant{}& C(1+t)^{-\frac{3}{2}}\big\{\|u_0\|^2_{L^2 \cap Z_1} + \big\||x| a_0\big\|^2_{L^1_x}\big\} \\ &+ C \sum_{j=1}^{2} \int_0^t (1+t-s)^{-\frac{3}{2}} \big\|\nu^{-1/2}\{\mathbf{I} - \mathbf{P}\} g_j(s)\big\|^2_{L^2 \cap Z_1}\, ds \\ &+ C\left\{\int_0^t (1+t-s)^{-\frac{3}{4}} \big\|\mathbf{P}_1 g_2(s)\big\|_{L^2 \cap Z_1}\, ds\right\}^2. \end{aligned} \tag{5.6}$$

As in [8] or in [27, Lemma 2.6], it is straightforward to verify

$$\big\|\nu^{-1/2}\{\mathbf{I} - \mathbf{P}\} g_j(t)\big\|_{L^2 \cap Z_1} + \big\|\mathbf{P}_1 g_2(t)\big\|_{L^2 \cap Z_1} \leqslant C\mathcal{E}_{N,\ell}(t), \tag{5.7}$$

where we notice $N \geqslant 4$ and $\ell \geqslant 2$. By (5.1), one has

$$\mathcal{E}_{N,\ell}(t) \leqslant (1+t)^{-3/2} X_{N,\ell}(t), \tag{5.8}$$

$0 \leqslant t < T$. With this, it follows from (5.6) and (5.7) that

$$\big\|\mathbf{P}_1 u(t)\big\|^2 + \big\|\nabla_x \Delta_x^{-1} \mathbf{P}_0 u(t)\big\|^2 \leqslant C(1+t)^{-\frac{3}{2}}\big\{\|u_0\|^2_{L^2 \cap Z_1} + \big\||x| a_0\big\|^2_{L^1_x} + X_{N,\ell}(t)^2\big\}. \tag{5.9}$$

Here, we used that $X_{N,\ell}(t)$ is nondecreasing in t. By the Gronwall inequality, (5.4) together with (5.9) yields

$$\mathcal{E}_{N,\ell}(t) \leqslant C(1+t)^{-\frac{3}{2}}\left\{\mathcal{E}_{N,\ell}(0) + \|u_0\|^2_{L^2\cap Z_1} + \big\||x|a_0\big\|^2_{L^1_x} + X_{N,\ell}(t)^2\right\},$$

which further implies

$$\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{3}{2}}\mathcal{E}_{N,\ell}(s) \leqslant C\left\{\mathcal{E}_{N,\ell}(0) + \|u_0\|^2_{Z_1} + \big\||x|a_0\big\|^2_{L^1_x} + X_{N,\ell}(t)^2\right\}. \tag{5.10}$$

This proves (5.2) corresponding to the right-hand first part of (5.1) for $X_{N,\ell}(t)$.

To deduce an estimate on the second term on the right-hand side of (5.1), we begin with (4.27). Notice $\mathcal{E}^{\mathrm{h}}_{N,\ell}(t) \leqslant C\mathcal{D}_{N,\ell}(t)$ by comparing (4.25) with (4.12), where the identity $\|\nabla_x^2\phi\| = \|\Delta_x\phi\| = \|a\|$ was used. Then, (4.27) implies

$$\frac{d}{dt}\mathcal{E}^{\mathrm{h}}_{N,\ell}(t) + \kappa\mathcal{E}^{\mathrm{h}}_{N,\ell}(t) \leqslant C\|\nabla_x \mathbf{P}u\|^2. \tag{5.11}$$

Similar to that of deducing (5.6) from Theorem 3.1, one has

$$\begin{aligned}\big\|\nabla_x\mathbf{P}u(t)\big\|^2 \leqslant{}& C(1+t)^{-\frac{5}{2}}\left\{\|u_0\|^2_{Z_1} + \|\nabla_x u_0\|^2 + \big\||x|a_0\big\|^2_{L^1_x}\right\}\\ &+ C\sum_{j=1}^{2}\int_0^t (1+t-s)^{-\frac{5}{2}}\left\{\big\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}g_j(s)\big\|^2_{Z_1}\right.\\ &\left.+\big\|\nu^{-1/2}\nabla_x\{\mathbf{I}-\mathbf{P}\}g_j(s)\big\|^2\right\}ds\\ &+ C\left\{\int_0^t (1+t-s)^{-\frac{5}{4}}\left(\big\|\mathbf{P}_1 g_2(s)\big\|_{Z_1} + \big\|\nabla_x\mathbf{P}_1 g_2(s)\big\|\right)ds\right\}^2.\end{aligned} \tag{5.12}$$

Once again, it is straightforward to verify

$$\begin{aligned}&\big\|\nu^{-1/2}\{\mathbf{I}-\mathbf{P}\}g_j(t)\big\|_{Z_1} + \big\|\nu^{-1/2}\nabla_x\{\mathbf{I}-\mathbf{P}\}g_j(t)\big\| + \big\|\mathbf{P}_1 g_2(t)\big\|_{Z_1} + \big\|\nabla_x\mathbf{P}_1 g_2(t)\big\|\\ &\quad\leqslant C\mathcal{E}_{N,\ell}(t).\end{aligned} \tag{5.13}$$

Similarly as before, by using (5.8) and

$$\int_0^t (1+t-s)^{-\frac{5}{2}}(1+s)^{-3}\,ds \leqslant C(1+t)^{-\frac{5}{2}},$$

$$\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{3}{2}}\,ds \leqslant C(1+t)^{-\frac{5}{4}},$$

it follows from (5.12) and (5.13) that

$$\big\|\nabla_x\mathbf{P}u(t)\big\|^2 \leqslant C(1+t)^{-\frac{5}{2}}\left\{\|u_0\|^2_{L^2_\xi(H^1_x)\cap Z_1} + \big\||x|a_0\big\|^2_{L^1_x} + X_{N,\ell}(t)^2\right\},$$

which together with (5.11) gives

$$\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{5}{2}}\mathcal{E}^{\mathrm{h}}_{N,\ell}(s)\leqslant C\left\{\mathcal{E}_{N,\ell}(0)+\|u_0\|^2_{Z_1}+\left\||x|a_0\right\|^2_{L^1_x}+X_{N,\ell}(t)^2\right\}. \tag{5.14}$$

Therefore, (5.2) holds by combining (5.10) and (5.14). This completes the proof of Lemma 5.1. □

Proof of Theorem 1.1. We first fix $N\geqslant 4$, $\ell\geqslant 2$, $\lambda>0$ and $0\leqslant\theta\leqslant 1/4$. The local existence and uniqueness of the solution $u(t,x,\xi)$ to the Cauchy problem (1.4)–(1.6) can be proved in terms of the energy functional $\mathcal{E}_{N,\ell}(t)$ given by (4.11), and the details are omitted for simplicity; see [16]. Then, one only has to obtain the uniform-in-time estimates over $0\leqslant t<T$ with $0<T\leqslant\infty$. In fact, by the continuity argument, Lemma 5.1 implies that under the *a priori* assumptions (**A1**) and (**A2**),

$$X_{N,\ell}(t)\leqslant C\left\{\mathcal{E}_{N,\ell}(0)+\|u_0\|^2_{Z_1}+\left\||x|a_0\right\|^2\right\},\quad 0\leqslant t<T, \tag{5.15}$$

provided that $\mathcal{E}_{N,\ell}(0)+\|u_0\|^2_{Z_1}+\||x|a_0\|^2$ is sufficiently small. The rest is to justify that the *a priori* assumptions (**A1**) and (**A2**) can be closed; see (4.5) and (4.26). Recall (1.9) and notice

$$\mathcal{E}_{N,\ell}(0)^{1/2}+\|u_0\|_{Z_1}+\left\||x|a_0\right\|\leqslant C\left\{\sum_{|\alpha|+|\beta|\leqslant N}\left\|w_\ell(0,\xi)\partial^\alpha_\beta u_0\right\|+\left\|\left(1+|x|\right)u_0\right\|_{Z_1}\right\}.$$

Here, in the above inequality, we used $\|\nabla_x\phi_0\|\leqslant C\|a_0\|^{2/3}_{L^1_x}\|a_0\|^{1/3}_{L^2_x}$ and

$$\left\|\nabla^2_x\phi_0\right\|_{H^{N-1}}=\left\|\nabla^2_x\Delta^{-1}_x a_0\right\|_{H^{N-1}}\leqslant C\|a_0\|_{H^{N-1}}.$$

Then, since $0<\theta\leqslant 1/4$, (**A1**) and (**A2**) directly follow from (5.15) together with (5.1), (4.11) and (4.25) as well as smallness of $\mathcal{E}_{N,\ell}(0)+\|u_0\|^2_{Z_1}+\||x|a_0\|^2$. Therefore, the uniform-in-time estimate (5.15) holds true for any $0\leqslant t<T$ as long as

$$\sum_{|\alpha|+|\beta|\leqslant N}\left\|w_\ell(0,\xi)\partial^\alpha_\beta u_0\right\|+\left\|\left(1+|x|\right)u_0\right\|_{Z_1}$$

is sufficiently small. Then, the global existence follows, and (1.10) holds from (5.15) by comparing (1.8) and (4.11). This completes the proof of Theorem 1.1. □

Acknowledgments

The research of the first author was supported by the General Research Fund (Project No.: CUHK 400511) from RGC of Hong Kong. The research of the second author was supported by the General Research Fund of Hong Kong, CityU No. 103108, and the Lou Jia Shan Scholarship programme of Wuhan University. And research of the third author was supported by two grants from the National Natural Science Foundation of China under contracts 10871151 and 10925103 respectively. This work is also supported by "The Fundamental Research Funds for the Central Universities".

References

[1] L. Bernis, L. Desvillettes, Propagation of singularities for classical solutions of the Vlasov–Poisson–Boltzmann equation, Discrete Contin. Dyn. Syst. 24 (2009) 13–33.
[2] C. Cercignani, R. Illner, M. Pulvirenti, The Mathematical Theory of Dilute Gases, Appl. Math. Sci., vol. 106, Springer-Verlag, New York, 1994.
[3] L. Desvillettes, J. Dolbeault, On long time asymptotics of the Vlasov–Poisson–Boltzmann equation, Comm. Partial Differential Equations 16 (1995) 451–489.

[4] L. Desvillettes, C. Villani, On the trend to global equilibrium for spatially inhomogeneous kinetic systems: The Boltzmann equation, Invent. Math. 159 (2) (2005) 245–316.
[5] R.-J. Duan, Dissipative property of the Vlasov–Maxwell–Boltzmann system with a uniform ionic background, SIAM J. Math. Anal. 43 (6) (2011) 2732–2757.
[6] R.-J. Duan, On the Cauchy problem for the Boltzmann equation in the whole space: Global existence and uniform stability in $L^2_\xi(H^N_x)$, J. Differential Equations 244 (2008) 3204–3234.
[7] R.-J. Duan, R.M. Strain, Optimal large-time behavior of the Vlasov–Maxwell–Boltzmann system in the whole space, Comm. Pure Appl. Math. 64 (11) (2011) 1497–1546.
[8] R.-J. Duan, R.M. Strain, Optimal time decay of the Vlasov–Poisson–Boltzmann system in $\mathbb{R}^3$, Arch. Ration. Mech. Anal. 199 (2010) 291–328.
[9] R.-J. Duan, T. Yang, Stability of the one-species Vlasov–Poisson–Boltzmann system, SIAM J. Math. Anal. 41 (6) (2010) 2353–2387.
[10] R.-J. Duan, T. Yang, H.-J. Zhao, The Vlasov–Poisson–Boltzmann system for soft potentials, preprint, 2011.
[11] R.T. Glassey, The Cauchy Problem in Kinetic Theory, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1996.
[12] R. Glassey, W. Strauss, Decay of the linearized Boltzmann–Vlasov system, Transport Theory Statist. Phys. 28 (1999) 135–156.
[13] R. Glassey, W. Strauss, Perturbation of essential spectra of evolution operators and the Vlasov–Poisson–Boltzmann system, Discrete Contin. Dyn. Syst. 5 (1999) 457–472.
[14] Y. Guo, The Boltzmann equation in the whole space, Indiana Univ. Math. J. 53 (2004) 1081–1094.
[15] Y. Guo, The Vlasov–Maxwell–Boltzmann system near Maxwellians, Invent. Math. 153 (2003) 593–630.
[16] Y. Guo, The Vlasov–Poisson–Boltzmann system near Maxwellians, Comm. Pure Appl. Math. 55 (9) (2002) 1104–1135.
[17] Y. Guo, J. Jang, Global Hilbert expansion for the Vlasov–Poisson–Boltzmann system, Comm. Math. Phys. 299 (2010) 469–501.
[18] S. Kawashima, Systems of a hyperbolic–parabolic composite type, with applications to the equations of magnetohydrodynamics, thesis, Kyoto University, 1983.
[19] S. Kawashima, The Boltzmann equation and thirteen moments, Japan J. Appl. Math. 7 (1990) 301–320.
[20] L. Li, On the trend to equilibrium for the Vlasov–Poisson–Boltzmann equation, J. Differential Equations 244 (2008) 1467–1501.
[21] T.-P. Liu, T. Yang, S.-H. Yu, Energy method for the Boltzmann equation, Phys. D 188 (3–4) (2004) 178–192.
[22] T.-P. Liu, S.-H. Yu, Boltzmann equation: micro–macro decompositions and positivity of shock profiles, Comm. Math. Phys. 246 (1) (2004) 133–179.
[23] S. Mischler, On the initial boundary value problem for the Vlasov–Poisson–Boltzmann system, Comm. Math. Phys. 210 (2000) 447–466.
[24] R.M. Strain, The Vlasov–Maxwell–Boltzmann system in the whole space, Comm. Math. Phys. 268 (2) (2006) 543–567.
[25] R.M. Strain, Y. Guo, Exponential decay for soft potentials near Maxwellian, Arch. Ration. Mech. Anal. 187 (2008) 287–339.
[26] S. Ukai, On the existence of global solutions of mixed problem for non-linear Boltzmann equation, Proc. Japan Acad. 50 (1974) 179–184.
[27] S. Ukai, T. Yang, The Boltzmann equation in the space $L^2 \cap L^\infty_\beta$: Global and time-periodic solutions, Anal. Appl. (Singap.) 4 (2006) 263–310.
[28] C. Villani, Hypocoercivity, Mem. Amer. Math. Soc. 202 (2009), iv+141.
[29] T. Yang, H.-J. Yu, Optimal convergence rates of classical solutions for Vlasov–Poisson–Boltzmann system, Comm. Math. Phys. 301 (2011) 319–355.
[30] T. Yang, H.-J. Yu, H.-J. Zhao, Cauchy problem for the Vlasov–Poisson–Boltzmann system, Arch. Ration. Mech. Anal. 182 (2006) 415–470.
[31] T. Yang, H.-J. Zhao, Global existence of classical solutions to the Vlasov–Poisson–Boltzmann system, Comm. Math. Phys. 268 (2006) 569–605.
[32] M. Zhang, Stability of the Vlasov–Poisson–Boltzmann system in $\mathbb{R}^3$, J. Differential Equations 247 (2009) 2027–2073.